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Stochastic orders in dynamic reinsurance markets

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Abstract. We consider a dynamic reinsurance market, where the traded risk process is driven by a compound Poisson process and where claim amounts are unbounded. These markets are known to be incomplete, and there are typically infinitely many martingale measures. In this case, no-arbitrage pricing theory can typically only provide wide bounds on prices of reinsurance claims. Optimal martingale measures such as the minimal martingale measure and the minimal entropy martingale measure are determined, and some comparison results for prices under different martingale measures are provided. This leads to a simple stochastic ordering result for the optimal martingale measures. Moreover, these optimal martingale measures are compared with other martingale measures that have been suggested in the literature on dynamic reinsurance markets.

Key words: Compound Poisson process, change of measure, minimal martingale measure, minimal entropy martingale measure, convex order, cut criterion, stop-loss contract

JEL Classification: G10

Mathematics Subject Classification (2000): 62P05, 60J75, 60G44

1 Introduction

Dynamic reinsurance markets have been studied in a continuous time framework using no-arbitrage conditions by Sondermann (1991) and Delbaen and Haezendonck (1989), among others. The main idea is to allow for dynamic rebalancing of proportional reinsurance covers, which is obtained by assuming that some process related to an *insurance risk process*, defined as accumulated premiums minus claims of some insurance business, is tradeable and that positions can be rebalanced continuously. This implies that reinsurers can change dynamically the amount of insurance business that they have accepted. In this way, the insurance risk process

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is viewed as a traded security, and this imposes no-arbitrage bounds on premiums for other reinsurance contracts such as stop-loss contracts.

This paper studies the situation where the traded index X, which is defined as claims less premiums on some insurance portfolio, is driven by a compound Poisson process. Thus, $-X$ is the so-called insurance risk process. This leads to an incomplete market with two traded assets, a savings account and the price process X. On the other hand, if the underlying insurance claims are described by an absolutely continuous distribution, there are essentially infinitely many sources of risk. As a consequence of this incompleteness, contingent claims (reinsurance contracts) cannot be priced uniquely by using no-arbitrage theory alone. In particular, there will be infinitely many martingale measures for X. For most contracts, different measures will lead to different prices, and it is not clear which measure one should apply. In this setting, we determine two optimal martingale measures known from the literature on incomplete financial markets, the minimal martingale measure and the minimal entropy martingale measure. These are candidate measures that can be used for pricing or valuation, see e.g., Schweizer (2001) and Grandits and Rheinländer (2002). The main result of the present paper is a criterion for the ordering of prices under two given martingale measures, which shows that these two optimal martingale measures are ordered in the so-called convex order. We show that for any convex function Φ, the price of the contract with payoff $\Phi(X_T)$ is smaller under the minimal martingale measure than under the minimal entropy martingale measure. For example, this is relevant for a stop-loss reinsurance contract $\Phi(X_T) = (X_T - K)^+$, which covers claims above some level (the sum of the premiums and the level K). This shows that the minimal entropy martingale measure in our model can be viewed as a more conservative attitude to risk than the minimal martingale measure.

The paper is organized as follows. Sect. 2 contains a review of existing results on dynamic reinsurance markets, and Sect. 3 introduces the basic model for the traded price process. Sect. 4 lists martingale measures suggested by Delbaen and Haezendonck (1989) and the minimal martingale measure and the minimal entropy martingale measure. The latter two are compared in Sect. 5. In Sect. 6, we give an ordering result for martingale measures and a comparison of the various martingale measures. Numerical results are presented for the case with exponentially distributed claims. The Appendix contains an extension to the time-inhomogeneous case, which also allows for a term driven by a Brownian motion. The results are related to a recent comparison result obtained by Henderson and Hobson (2003) within exponential jump-diffusion models.

2 Existing literature on dynamic reinsurance markets

We briefly review the results of Sondermann (1991) and Delbaen and Haezendonck (1989); see also Embrechts (2000) and Møller (2002). Let U_t be the accumulated claims during $[0,t]$ from some insurance business and let $\widetilde{p} = (\widetilde{p}_t)_{0\leq t\leq T}$ be a predictable process related to the premiums on this business. We assume that the interest rate on the market is constant and equal to 0. Define the (discounted) traded process by $X = U - \widetilde{p}$ and consider a savings account with (discounted) price process $X^\circ = 1$.

Sondermann (1991) takes $\widetilde{p}_t$ to be the premiums paid during $[0, t]$. In this case, X_t corresponds to the value at time t of an account where claims are added and premiums subtracted as they incur. Reinsurers can now participate in the risk by trading the asset X. If premiums are paid continuously at a fixed rate κ, we get $\widetilde{p}_t = \kappa t$. In particular, a position of $-\xi$ units during some interval $(t, t+h)$ will lead to the trading gains

$$-\xi(X_{t+h} - X_t) = \xi\kappa h - \xi(U_{t+h} - U_t). \tag{2.1}$$

If $\xi \in (0,1)$, this means that the reinsurer receives a fraction ξ of the premiums and covers a fraction ξ of the claims during $(t, t+h)$. Thus, the gains (2.1) correspond to the amount received by a reinsurer who has provided a so-called proportional reinsurance cover for the underlying insurance portfolio. Sondermann (1991) studied this model for a dynamic market for proportional reinsurance contracts, and demonstrated that traditional reinsurance contracts such as stop-loss contracts could be viewed as contingent claims, which should be priced so that no arbitrage possibilities arise.

Delbaen and Haezendonck (1989) applied an alternative definition of the traded process X and defined $-\widetilde{p}_t$ as the premium at which the direct insurer can sell the remaining risk $U_T - U_t$ on the reinsurance market, such that X_t would represent the insurer's liabilities at time t. More precisely, X_t is equal to the claims U_t incurred before time t, added a premium $-\widetilde{p}_t$ for the remaining claims $U_T - U_t$ during $(t, T]$. If the direct insurer receives continuously paid premiums at the rate κ and this rate coincides with the one charged by the reinsurers, we get $-\widetilde{p}_t = \kappa(T-t)$, which differs from Sondermann's choice only by the constant κT. Delbaen and Haezendonck (1989) considered a compound Poisson process U and studied equivalent measures Q, where U is also a Q-compound Poisson process. They then determined a predictable premium process $-\widetilde{p}$ by requiring that X be a Q-martingale. This approach, which guaranteed that no arbitrage possibilities could arise from trading in X, was used to recover traditional actuarial valuation principles by considering specific martingale measures.

In practice, reinsurance contracts are agreements between an insurer and a reinsurer, and these contracts are typically not traded on stock exchanges. In addition, these agreements are subject to credit risk, since reinsurance companies might default as a result of extreme losses. Thus, the current setting of a dynamic reinsurance market, which moreover deals with one insurance portfolio only, should clearly be viewed as an idealization.

3 Model and notation

We consider the discounted traded price process

$$X_t = X_0 + \sum_{j=1}^{N_t} Y_j - \kappa t = X_0 + U_t - \kappa t, \tag{3.1}$$

where N is a homogeneous Poisson process with intensity λ, and where N is independent of the sequence $Y_1, Y_2, \ldots$ of i.i.d. random variables with distribution function G on $(0, \infty)$. The processes are defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with the natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ associated with X; T is a fixed finite time horizon. In Appendix 6.4, the framework is extended by allowing for an inhomogeneous compound Poisson process and a term driven by a standard Brownian motion.

Denote by μ_k the k'th moment (under the measure P) of the claim-size distribution G. We require that the premium rate is given by $\kappa = (1+\vartheta)\,\lambda\,\mu_1 > \lambda\,\mu_1$, where $\vartheta > 0$ is the strictly positive deterministic *safety-loading* parameter. We assume that $\mathrm{E}[e^{\eta Y_1}] < \infty$ for η in some open interval containing 0, so that in particular $\mu_k < \infty$ for all $k > 0$.

Change of measure for compound Poisson processes
Consider measures Q defined by $\frac{dQ}{dP} = Z_T$, with density process $Z_t = \mathrm{E}[Z_T|\mathcal{F}_t]$ given by

$$Z_t = e^{-\lambda t \mathrm{E}[\phi(Y_1)]} e^{\sum_{j=1}^{N_t} \log(1+\phi(Y_j))}, \tag{3.2}$$

where ϕ is some deterministic function with $\phi(y) > -1$, for all $y > 0$, and $\mathrm{E}[\phi(Y_1)] < \infty$. Delbaen and Haezendonck (1989) showed that the process $U_t = \sum_{j=1}^{N_t} Y_j$ is a compound Poisson process under Q with parameters

$$\lambda^Q = \lambda(1+\mathrm{E}[\phi(Y_1)]) = (1+\widetilde{\phi})\lambda, \tag{3.3}$$

$$G^Q(dy) = \frac{1+\phi(y)}{1+\mathrm{E}[\phi(Y_1)]} G(dy) = \frac{1+\phi(y)}{1+\widetilde{\phi}} G(dy) = \frac{\lambda}{\lambda^Q}(1+\phi(y))G(dy). \tag{3.4}$$

For reasons of completeness, we show that this is indeed the case by simply calculating the characteristic function of the increment $U_t - U_\tau$ given $\mathcal{F}_\tau$, for fixed $\tau < t \leq T$. Recall that the characteristic function under P, given $\mathcal{F}_\tau$, of the compound Poisson variable $U_t - U_\tau = \sum_{j=N_\tau+1}^{N_t} Y_j$ is

$$\mathrm{E}\left[e^{is(U_t-U_\tau)} \middle| \mathcal{F}_\tau \right] = e^{\lambda(t-\tau)\left(\mathrm{E}[e^{isY_1}]-1\right)},$$

which involves λ and the characteristic function of the random variable Y_1 with distribution G. Under Q, the corresponding characteristic function is

$$\begin{aligned}
&\mathrm{E}_Q\left[e^{is(U_t-U_\tau)} \middle| \mathcal{F}_\tau \right] \\
&= \mathrm{E}\left[\frac{Z_t}{Z_\tau} e^{is(U_t-U_\tau)} \middle| \mathcal{F}_\tau \right] \\
&= \mathrm{E}\left[e^{-\lambda(t-\tau)\mathrm{E}[\phi(Y_1)]} e^{\sum_{j=N_\tau+1}^{N_t} \log(1+\phi(Y_j))} e^{is\sum_{j'=N_\tau+1}^{N_t} Y_{j'}} \middle| \mathcal{F}_\tau \right] \\
&= e^{-\lambda(t-\tau)\mathrm{E}[\phi(Y_1)]} \mathrm{E}\left[\prod_{j=1}^{N_t-N_\tau} \left(\frac{e^{isY_j}(1+\phi(Y_j))}{1+\mathrm{E}[\phi(Y_1)]} (1+\mathrm{E}[\phi(Y_1)]) \right) \right] \\
&= e^{\lambda(1+\mathrm{E}[\phi(Y_1)])(t-\tau)(\mathrm{E}_Q[e^{isY_1}]-1)}.
\end{aligned} \tag{3.5}$$

The first equality follows by using the definition of the measure Q and the abstract Bayes formula, and the second equality follows by inserting the expression (3.2) for the density process Z. In the third equality, we have used the fact that N has independent and stationary increments under P and that $Y_1, Y_2, \dots$ are i.i.d. and independent of N under P. In the last equality, which follows by using the properties of N and $Y_1, Y_2, \dots$ under P, we have moreover introduced the notation

$$\mathrm{E}_Q[e^{isY_1}] = \int_{(0,\infty)} e^{isy} \frac{1+\phi(y)}{1+\mathrm{E}[\phi(Y_1)]} G(dy) = \int_{(0,\infty)} e^{isy} G^Q(dy),$$

where G^Q is defined by (3.4). This shows that (3.5) is the characteristic function of a compound Poisson variable with Poisson parameter $\lambda^Q(t-\tau)$ and claim size distribution G^Q. Thus, U is indeed a Q-compound Poisson process with parameters (3.3) and (3.4). We let $\mu^Q = \mathrm{E}_Q[Y_1]$ and note that the process $U_t - \lambda^Q \mu^Q t$ is a Q-martingale.

4 Equivalent martingale measures

We can rewrite X as $X = X_0 + M + A$, where M is a P-martingale given by

$$M_t = \sum_{j=1}^{N_t} Y_j - \lambda \mu_1 t, \tag{4.1}$$

and where A is a deterministic process defined by $A_t = (\lambda\mu_1 - \kappa)t$. It is well-known that there can be many martingale measures for X, see Delbaen and Haezendonck (1989). A martingale measure can, loosely speaking, be defined by changing the intensity λ for the occurrence of claims, by changing the distribution G for the claim sizes, or by combinations of these two methods. In the literature on incomplete financial markets, one can find notions such as the minimal martingale measure and the minimal entropy martingale measure. In this section, we list these martingale measures for the process X. These results are well-known and can essentially be compiled from various other papers on pricing of options on assets driven by jump-diffusions; for related results on log-Lévy processes, see e.g., Chan (1999) and Henderson and Hobson (2003).

Here we give conditions on ϕ for a measure Q with density process (3.2) to be a martingale measure. Since $X_t = X_0 + (U_t - \lambda^Q\mu^Q t) + (\lambda^Q\mu^Q - \kappa)t$, we see that X is a Q-martingale if $\kappa = \mu^Q \lambda^Q$. To relate this to ϕ, we rewrite this as

$$\lambda \int_{(0,\infty)} y(1+\phi(y))G(dy) - (1+\vartheta)\mu_1\lambda = 0. \tag{4.2}$$

We shall also refer to (4.2) as the *martingale equation*, and ϕ is called the *kernel*.

4.1 Examples of martingale measures

We consider three examples of martingale measures Q^{β}, proposed by Delbaen and Haezendonck (1989), with densities of the form

$$\frac{dQ^{\beta}}{dP} = C \exp\left(\sum_{j=1}^{N_T} \beta(Y_j)\right),$$

which means that $\beta(y) = \log(1 + \phi(y))$ in the notation used above.

One example is to take β constant, $\beta(y) = \log(1+\zeta)$, $\zeta > -1$. In this case, the martingale Eq. (4.2) is solved by $\zeta = \vartheta$. Thus, the Poisson intensity is changed to $(1+\vartheta)\lambda$, see (3.3), while the distribution (3.4) of the claims remains unchanged. We refer to this measure as $Q^{(1)}$ and the parameters are $(\lambda^{(1)}, G^{(1)}) = ((1+\vartheta)\lambda, G)$.

Another example is $\beta(y) = \log(1 + b(y - \mu_1))$, for some $b > 0$. In this case, the martingale equation becomes

$$\lambda \int y(1 + b(y - \mu_1))G(dy) = \lambda(1+\vartheta)\mu_1,$$

which shows that this martingale measure $Q^{(2)}$ is determined by $b = \vartheta\mu_1/(\mu_2 - (\mu_1)^2)$. Note that the condition $\phi(y) > -1$ for all y is only satisfied if $\frac{\vartheta(\mu_1)^2}{\mu_2-(\mu_1)^2} < 1$, which means that the safety-loading ϑ should not be too big. Under this martingale measure, the Poisson parameter is unchanged, i.e. $\lambda^{(2)} = \lambda$, whereas the distribution for the claim amounts is given by $G^{(2)}(dy) = \left(1 + \frac{\vartheta\mu_1}{\mu_2-(\mu_1)^2}(y - \mu_1)\right) G(dy)$.

A third example is $\beta(y) = \rho y - \log(\int e^{\rho y} G(dy))$, for some $\rho > 0$. Here, $1 + \phi(y) = e^{\rho y}/\int e^{\rho y} G(dy)$, which can be inserted in the martingale equation to determine a martingale measure $Q^{(3)}$. Under this measure, the Poisson parameter $\lambda^{(3)}$ is again unaffected by the change of measure and equal to λ, whereas the distribution of the claim amounts changes to $G^{(3)}(dy) = (1 + \phi(y))G(dy)$.

4.2 The minimal martingale measure

Define a martingale $\widehat{Z}$ via $d\widehat{Z}_t = -\widehat{Z}_t \alpha dM_t$, with $\widehat{Z}_0 = 1$, where $\alpha = -\vartheta\mu_1/\mu_2$ and where M is defined by (4.1). Since $\alpha < 0$, it can be verified that

$$\widehat{Z}_t = e^{\alpha\lambda\mu_1 t} e^{\sum_{j=1}^{N_t} \log(1-\alpha Y_j)}, \tag{4.3}$$

which is of the form (3.2) with $\phi(y) = -\alpha y$. Since $\mathrm{E}[\widehat{Z}_T] = 1$, we can define an equivalent martingale measure $\widehat{P}$ via $\frac{d\widehat{P}}{dP} = \widehat{Z}_T$. The measure $\widehat{P}$ is known as the *minimal martingale measure*, see Föllmer and Schweizer (1991), and it is well-known that this measure is in general only a *signed measure* for discontinuous processes; see also Schweizer (1995). However, in our model the assumption $\vartheta > 0$ guarantees that $\alpha < 0$, which implies that $\widehat{P}$ is indeed a true probability measure.

The dynamics of X under the minimal martingale measure can now be found via the change of measure result recalled in Sect. 3 by comparing (4.3) and (3.2).

Under the minimal martingale measure $\widehat{P}$, the Poisson intensity is $\widehat{\lambda} = (1-\alpha\,\mu_1)\lambda$, which exceeds the intensity λ under P, and the claim size distribution becomes $\widehat{G}(dy) := \frac{1-\alpha\, y}{1-\alpha\,\mu_1} G(dy)$. It follows that $\mu_1 = \mathrm{E}[Y_1] \leq \mathrm{E}_{\widehat{P}}[Y_1] = \widehat{\mu}_1$. Thus, the change of measure from P to the minimal martingale measure $\widehat{P}$ increases the Poisson intensity as well as expected claim sizes. (To see that $\widehat{P}$ is a martingale measure, check that the martingale equation is satisfied for $\phi(y) = -\alpha y$, where $\alpha = -\vartheta\mu_1/\mu_2$.)

4.3 The minimal entropy martingale measure

An equivalent martingale measure is called the *minimal entropy martingale measure* if it minimizes the so-called relative entropy with respect to P. It follows e.g., from Grandits and Rheinländer (2002) that the density of the minimal entropy martingale measure $\overline{P}$ is of the form $\frac{d\overline{P}}{dP} = C \exp\left(\int_0^T \eta_s\, dX_s\right)$, for some constant C and some integrable process η.

Consider a measure $\overline{P}^{\eta}$ on the above form, where η is some constant. (For most choices of η, $\overline{P}^{\eta}$ will not be a martingale measure.) Using the simple expression for X, we can rewrite the density as (here C is a constant which changes from expression to expression):

$$\begin{aligned} \frac{d\overline{P}^{\eta}}{dP} &= C \exp\left(\int_0^T \eta\, d(M_s + A_s)\right) \\ &= C \exp\left(\int_0^T \eta\, dM_s\right) = C \exp\left(\sum_{j=1}^{N_T} \log e^{\eta Y_j}\right). \end{aligned} \tag{4.4}$$

Thus, for each η, we have an equivalent measure $\overline{P}^{\eta}$ with density (4.4) with respect to P. Note that the structure of (4.4) is similar to (3.2), so that we can immediately derive the $\overline{P}^{\eta}$-properties of the compound Poisson process U which drives X. Let

$$\psi(\eta) = \int_{(0,\infty)} e^{\eta y} G(dy) = \mathrm{E}\left[e^{\eta Y_1}\right].$$

By comparing (3.2) and (4.4), we see that the parameters are $\overline{\lambda} = \psi(\eta)\lambda$ and $\overline{G}(dy) = \frac{e^{\eta y}}{\psi(\eta)} G(dy)$. The minimal entropy martingale measure is uniquely determined via the solution η to the martingale Eq. (4.2). By rewriting this equation for η as

$$\lambda \int_{(0,\infty)} y(e^{\eta y} - 1) G(dy) - \vartheta\lambda\mu_1 = 0, \tag{4.5}$$

we see that $\eta > 0$, since $\vartheta > 0$.

The case of gamma distributed claims
Assume as an example that G is the gamma distribution with parameters (β, δ), i.e.

that $G(dy) = g_{(\beta,\delta)}(y)dy$, where

$$g_{(\beta,\delta)}(dy) = \frac{\delta^\beta}{\Gamma(\beta)} y^{\beta-1} e^{-\delta y}. \tag{4.6}$$

Then, it follows that

$$\int_{(0,\infty)} y e^{\eta y} G(dy) = \frac{\beta\delta^\beta}{(\delta-\eta)^{\beta+1}},$$

for $\eta < \delta$. In particular, for β integer valued, solving (4.5) involves finding a real root in a polynomial of order $\beta + 1$.

5 Comparisons of optimal martingale measures

A natural question is whether there is a systematic ordering of prices computed under the two optimal martingale measures introduced above. This question has recently been studied in a different context within an exponential jump-diffusion model by Henderson and Hobson (2003), among others, who gave a simple criterion for ordering of option prices under various martingale measures. Further comparison results on option prices in markets driven by semimartingales have been obtained by Gushchin and Mordecki (2002). For a comparison of our results with the ones obtained by Henderson and Hobson (2003), see the Appendix.

Let Q and $\widetilde{Q}$ be two martingale measures for X and let

$$X_T = X_0 + U_T - \lambda^Q \mu^Q T = X_0 + U_T - \lambda^{\widetilde{Q}} \mu^{\widetilde{Q}} T, \tag{5.1}$$

where U is a compound Poisson process with parameters (λ^Q, G^Q) and $(\lambda^{\widetilde{Q}}, G^{\widetilde{Q}})$ under Q and $\widetilde{Q}$, respectively. The kernel for the minimal martingale measure is $\phi^{\widehat{P}}(y) = -\alpha y$, and for the minimal entropy martingale measure, $\phi^{\overline{P}}(y) = \exp(\eta y) - 1$, where η solves the Eq. (4.5). The following result gives bounds on η:

Lemma 5.1 *The parameter η in the kernel $\phi^{\overline{P}}(y) = e^{\eta y} - 1$ is positive, and $\eta < -\alpha$.*

Proof We can realize this by a straightforward examination of (4.5). By adding and subtracting the term ηy, the integrand can be rewritten as $e^{\eta y} - 1 = \eta y + f(\eta, y)$, where f is defined by $f(\eta, y) = e^{\eta y} - (1 + \eta y)$, which is strictly positive for $\eta > 0$ and $y > 0$ (Taylor expansion of e^x at 0). Thus, the integral appearing in (4.5) becomes

$$\begin{aligned}\lambda \int_{(0,\infty)} y(e^{\eta y} - 1) G(dy) &= \eta\lambda \int_{(0,\infty)} y^2 G(dy) + \lambda \int_{(0,\infty)} y f(\eta, y) G(dy) \\ &= \eta\lambda\mu_2 + \lambda F(\eta),\end{aligned}$$

where $F(\eta) > 0$. By inserting this in (4.5), we get that $\eta\,(\lambda\mu_2) + \lambda F(\eta) = \vartheta\lambda\mu_1$, which shows that

$$\eta < \eta + \frac{\lambda F(\eta)}{\sigma^2 + \lambda\mu_2} = \frac{\vartheta\lambda\mu_1}{\sigma^2 + \lambda\mu_2} = -\alpha. \qquad \square$$

We can show that $\widehat{\lambda} \geq \overline{\lambda}$, i.e., the Poisson arrival intensity under the minimal martingale measure exceeds the one for the minimal entropy martingale measure. Using the definition of $\overline{\lambda}$, Taylor expansion for the exponential function and Tonelli's Theorem, we first rewrite the Poisson intensity under the minimal entropy martingale measure as

$$\begin{aligned}\overline{\lambda} &= \lambda \int_{(0,\infty)} e^{\eta y} G(dy) \\ &= \lambda \int_{(0,\infty)} \left(\sum_{m=0}^{\infty} \frac{(\eta y)^m}{m!} \right) G(dy) = \lambda \left(1 + \sum_{m=1}^{\infty} \eta^m \frac{\mu_m}{m!} \right). \end{aligned} \tag{5.2}$$

Here, η is found from the Eq. (4.5), which via similar calculations simplifies to

$$\vartheta \mu_1 = \int_{(0,\infty)} y(e^{\eta y} - 1) G(dy) = \sum_{m=1}^{\infty} \eta^m \frac{\mu_{m+1}}{m!}. \tag{5.3}$$

Under the minimal martingale measure, the Poisson intensity is $\widehat{\lambda} = \lambda(1 - \alpha\mu_1) = \lambda\left(1 + \vartheta \frac{\mu_1^2}{\mu_2}\right)$. By inserting (5.3) in the expression for $\widehat{\lambda}$, we get

$$\widehat{\lambda} = \lambda \left(1 + \frac{\mu_1}{\mu_2} \sum_{m=1}^{\infty} \eta^m \frac{\mu_{m+1}}{m!} \right),$$

which can now be compared directly with (5.2). To see that $\widehat{\lambda} \geq \overline{\lambda}$ it only remains to verify that $\mu_1 \, \mu_{m+1} \geq \mu_m \, \mu_2$, for $m = 1, 2, \ldots$. This follows by using the inequality $\mathrm{E}^*[Y_1^m] \geq \mathrm{E}^*[Y_1^{m-1}]\mathrm{E}^*[Y_1]$ under the measure $G^*(dy) = \frac{y}{\mu_1} G(dy)$. Since any martingale measure Q has the property $\kappa = \mu^Q \lambda^Q$, see Sect. 4, we have that $\widehat{\mu}\,\widehat{\lambda} = \overline{\mu}\,\overline{\lambda}$, which implies that $\widehat{\mu}_1 \leq \overline{\mu}_1$. Thus, the expected values in the claim size distributions are ordered.

6 Stochastic orders and optimal martingale measures

6.1 Some results from the theory on stochastic orders

This section reviews standard results from the theory on stochastic orders that will prove useful below; references are Müller and Stoyan (2002, Chapt. 1) and Shaked and Shanthikumar (1994, Chapt. 2). For applications of stochastic orders in actuarial science, see e.g., Goovaerts et al. (1984) and Kaas et al. (1994).

Consider two random variables Y and Z, with distribution functions F_Y and F_Z, respectively. The random variable Y is said to be *stochastically smaller* than Z if $F_Y(y) \geq F_Z(y)$ for all y. In this case we write $Y \preceq_d Z$ or $F_Y \preceq_d F_Z$; note than this is a condition on the distribution functions for Y and Z and that Y and Z need not be defined on the same probability space. It is not difficult to see that if Y and Z are non-negative and if $Y \preceq_d Z$, then $\mathrm{E}[Y^r] \leq \mathrm{E}[Z^r]$ for all $r \geq 0$.

The random variable Y is said to be smaller than Z in the *increasing convex order* (written $Y \preceq_c Z$ or $F_Y \preceq_c F_Z$) if

$$\mathrm{E}[(Y-x)^+] = \int_x^\infty (y-x)F_Y(dy) \leq \int_x^\infty (y-x)F_Z(dy) = \mathrm{E}[(Z-x)^+]$$

for all x. It follows by Jensen's inequality that $\mathrm{E}[Y] \preceq_c Y$, i.e. Y is larger in the increasing convex order than its expected value. It can be shown that $Y \preceq_c Z$ if and only if $\mathrm{E}[\Phi(Y)] \leq \mathrm{E}[\Phi(Z)]$ for all increasing convex functions Φ. If, moreover, $\mathrm{E}[Y] = \mathrm{E}[Z]$, then this inequality holds for all (not necessarily increasing) convex functions. In this situation, we simply say that Y is smaller than Z in the convex order.

Let Θ be another random variable and denote by $F_{Y,\theta}$ and $F_{Z,\theta}$ the conditional distributions of Y and Z given $\Theta = \theta$. If $F_{Y,\theta} \preceq_c F_{Z,\theta}$ for all θ, then $F_Y \preceq_c F_Z$. Thus, the (increasing) convex order is *closed under mixture*. If $Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$ are sequences of independent random variables such that $Y_j \preceq_c Z_j$ for $j = 1, 2, \ldots$ then $g(Y_1, \ldots, Y_m) \preceq_c g(Z_1, \ldots, Z_m)$ for any increasing, component-wise convex function $g : \mathbb{R}^m \mapsto \mathbb{R}$. In particular, this shows that $Y_1 + \ldots + Y_m \preceq_c Z_1 + \ldots + Z_m$, so that the (increasing) convex order is *closed under convolution*. If moreover each of the sequences $(Y_j)_{j\in\mathbb{N}}$ and $(Z_j)_{j\in\mathbb{N}}$ are i.i.d. and if N and M are two integer-valued non-negative random variables with $M \preceq_c N$ which are independent of the sequences of Y_j's and Z_j's, then

$$\sum_{j=1}^{M} Y_j \preceq_c \sum_{j=1}^{N} Z_j. \tag{6.1}$$

This shows that the (increasing) convex order is *closed under random summation*. For example, if M and N are Poisson variables with parameters λ_M and λ_N, then $M \preceq_c N$ if $\lambda_M \leq \lambda_N$. A useful sufficient criterion for the (increasing) convex ordering of two random variables Y and Z with distribution functions F_Y and F_Z is the so-called *cut criterion*: If $\mathrm{E}[Y] \leq \mathrm{E}[Z]$ and if there exists some finite ξ such that

$$F_Y(y) \leq F_Z(y) \text{ for all } y < \xi, \text{ and } F_Y(y) \geq F_Z(y) \text{ for all } y > \xi,$$

then $Y \preceq_c Z$. If the distribution functions F_Y and F_Z admit densities f_Z and f_Y with respect to some measure and have equal means, then the cut criterion is satisfied if for example the function $(f_Z - f_Y)$ has exactly two sign changes, with sign sequence $+$, $-$, $+$.

6.2 Main results: Convex ordering for martingale measures

We give a sufficient criterion for the convex ordering of the distribution of X_T for two equivalent martingale measures Q and $\widetilde{Q}$ with *deterministic* kernels ϕ^Q and $\phi^{\widetilde{Q}}$, respectively. Since X_T is determined by compound Poisson variables, we can apply the results on convex orders reviewed in the previous section. The parameters are

(λ^Q, G^Q) and $(\lambda^{\widetilde{Q}}, G^{\widetilde{Q}})$, and the means in the claim size distributions are μ_1^Q and $\mu_1^{\widetilde{Q}}$. Finally, F_Q and $F_{\widetilde{Q}}$ are the distribution functions of X_T under Q and $\widetilde{Q}$. Note that for any martingale measures Q and $\widetilde{Q}$, we have that $\mathrm{E}_Q[X_T] = \mathrm{E}_{\widetilde{Q}}[X_T] = X_0$, so that we do not need to distinguish between the increasing convex order and the convex order when comparing the distribution of X_T under different martingale measures. Here is the main result:

Theorem 6.1 *Consider equivalent martingale measures Q and $\widetilde{Q}$ with deterministic kernels ϕ^Q and $\phi^{\widetilde{Q}}$ and parameters (λ^Q, G^Q) and $(\lambda^{\widetilde{Q}}, G^{\widetilde{Q}})$. Let $v(y) = \phi^Q(y) - \phi^{\widetilde{Q}}(y)$, and assume that:*
1. $\mu_1^Q \geq \mu_1^{\widetilde{Q}}$ and $\lambda^{\widetilde{Q}} \mu_1^{\widetilde{Q}} = \lambda^Q \mu_1^Q$.
2. There exist constants $0 \leq y^1 \leq y^2 < \infty$, such that $v(y) \geq 0$ for $y \in (0, y^1) \cup (y^2, \infty)$ and $v(y) \leq 0$ for $y \in (y^1, y^2)$.
Then $F_{\widetilde{Q}} \preceq_c F_Q$, i.e. for any convex function Φ, $\mathrm{E}_{\widetilde{Q}}[\Phi(X_T)] \leq \mathrm{E}_Q[\Phi(X_T)]$.

Remark 6.2 The first condition guarantees that the means of the claim size distributions under Q and $\widetilde{Q}$ are ordered, and the second condition is needed in order to ensure that we can apply the cut criterion on certain transformed densities related to the two measures. Since Q and $\widetilde{Q}$ are both martingale measures, $\lambda^{\widetilde{Q}} \mu_1^{\widetilde{Q}} = \lambda^Q \mu_1^Q = (1 + \vartheta)\mu_1 \lambda = \kappa$. □

We postpone the proof of Theorem 6.1 to Sect. 6.4 below. Here, we formulate and prove instead Proposition 6.3 by using this theorem. Denote by $\widehat{F}$ and $\overline{F}$ the distribution functions of X_T under the minimal martingale measure $\widehat{P}$ and the minimal entropy martingale measure $\overline{P}$, respectively. In addition, we use the notation $\widehat{\mathrm{E}}$ and $\overline{\mathrm{E}}$ for the expectations $\mathrm{E}_{\widehat{P}}$ and $\mathrm{E}_{\overline{P}}$. The next result shows that $\widehat{F} \preceq_c \overline{F}$, which implies that the minimal entropy martingale measure represents a more conservative attitude to risk than the minimal martingale measure.

Proposition 6.3 *For any convex function Φ, the price of $\Phi(X_T)$ under the minimal martingale measure is smaller than the price of $\Phi(X_T)$ under the minimal entropy martingale measure, i.e. $\widehat{\mathrm{E}}[\Phi(X_T)] \leq \overline{\mathrm{E}}[\Phi(X_T)]$.*

Proof First recall that the kernels for the minimal martingale measure $\widehat{P}$ and the minimal entropy martingale measure $\overline{P}$ are indeed deterministic. Secondly, we know from above that $\widehat{\lambda}\widehat{\mu}_1 = \overline{\lambda}\overline{\mu}_1$, and that $\widehat{\mu}_1 \leq \overline{\mu}_1$. This establishes the first condition of Theorem 6.1. For the measures $(\overline{P}, \widehat{P})$, we have that $v(y) = e^{\eta y} - (1 - \alpha y)$ with $\eta < -\alpha$. In this case, Condition 2 is clearly satisfied with $y^1 = 0$ and y^2 given as the unique strictly positive solution to the equation $v(y) = 0$. This completes the proof. □

6.3 Optimal measures versus ad-hoc choices

In this section, we compare the measures determined by ad-hoc considerations in Sect. 4.1 with the two optimal martingale measures, see Table 6.2. We end the section by considering an example with exponentially distributed claims.

Table 1. Martingale measures and their parameters

Meas.	Poisson int.	Claim size distribution	$1+\phi(y)$
$Q^{(1)}$	$(1+\vartheta)\lambda$	$G(dy)$	$1+\vartheta$
$Q^{(2)}$	λ	$(1+\frac{\vartheta\mu_1}{\mu_2-(\mu_1)^2}(y-\mu_1))G(dy)$	$1+\frac{\vartheta\mu_1}{\mu_2-(\mu_1)^2}(y-\mu_1)$
$Q^{(3)}$	λ	$e^{\rho y}/(\int e^{\rho y'}G(dy'))G(dy)$	$e^{\rho y}/(\int e^{\rho y'}G(dy'))$
$\widehat{P}$	$(1+\vartheta\frac{\mu_1^2}{\mu_2})\lambda$	$(1+\frac{\vartheta\mu_1}{\mu_2+\vartheta(\mu_1)^2}(y-\mu_1))G(dy)$	$1+\frac{\vartheta\mu_1}{\mu_2+\vartheta(\mu_1)^2}(y-\mu_1)$
$\overline{P}$	$\lambda\int e^{\eta y'}G(dy')$	$e^{\eta y}/(\int e^{\eta y'}G(dy'))G(dy)$	$e^{\eta y}$

Table 2. Probability measures and their parameters under exponentially distributed claims

Measure	Poisson intensity	Claim size distr. (density)
P	λ	$g_{(1,\delta)}$
$Q^{(1)}$	$(1+\vartheta)\lambda$	$g_{(1,\delta)}$
$Q^{(2)}$	λ	$(1-\vartheta)g_{(1,\delta)}+\vartheta g_{(2,\delta)}$
$Q^{(3)}$	λ	$g_{(1,\delta/(1+\vartheta))}$
$\widehat{P}$	$(1+\vartheta/2)\lambda$	$\frac{1}{1+\vartheta/2}g_{(1,\delta)}+\frac{\vartheta/2}{1+\vartheta/2}g_{(2,\delta)}$
$\overline{P}$	$\lambda\sqrt{1+\vartheta}$	$g_{(1,\frac{\delta}{\sqrt{1+\vartheta}})}$

Proposition 6.4 *The martingale measures are ordered in the following way:*

$$F_{Q^{(1)}} \preceq_c F_{\widehat{P}} \preceq_c F_{\overline{P}} \preceq_c F_{Q^{(3)}}. \tag{6.2}$$

If $\vartheta(\mu_1)^2/(\mu_2-(\mu_1)^2) < 1$, *then* $Q^{(2)}$ *is well-defined and* $F_{Q^{(1)}} \preceq_c F_{\widehat{P}} \preceq_c F_{Q^{(2)}} \preceq_c F_{Q^{(3)}}$.

Proof As mentioned in Remark 6.2, the means in the compound Poisson parts under the various martingale measures coincide, such that the second part of Condition 1 in Theorem 6.1 will automatically be satisfied for any pair of martingale measures. We first verify the three ordering relations in (6.2) by showing in each case that the conditions of the theorem are satisfied:

$F_{Q^{(1)}} \preceq_c F_{\widehat{P}}$: Compare first the Poisson intensities under the two measures: Since $\lambda^{Q^{(1)}} = (1+\vartheta)\lambda \geq (1+\vartheta\frac{(\mu_1)^2}{\mu_2}) = \widehat{\lambda}$, we see that $\mu_1^{Q^{(1)}} \leq \widehat{\mu}_1$. This shows the first part of Condition 1. To check the second condition, note that $v(y) = |\alpha|y - \vartheta$, which clearly satisfies Condition 2.

$F_{\widehat{P}} \preceq_c F_{\overline{P}}$: This already follows from Proposition 6.3.

$F_{\overline{P}} \preceq_c F_{Q^{(3)}}$: The first condition of Theorem 6.1 is verified by noting that $\lambda^{Q^{(3)}} = \lambda \leq \overline{\lambda}$, and the second condition follows by noting that $v(y) = e^{\rho y}/(\int e^{\rho y'}G(dy')) - e^{\eta y}$, where $\rho > \eta$. Thus, $v(y)$ satisfies the second condition of the theorem. This completes the proof of (6.2). The condition on ϑ for the existence of $Q^{(3)}$ follows from Sect. 4.1, and the ordering relations for the measures $Q^{(i)}$, $i = 1, 2, 3$, and $\widehat{P}$ follow from calculations similar to the ones used for the proof of (6.2). □

The case of exponentially distributed claims

Consider the situation, where the claim size distribution G under P is exponential

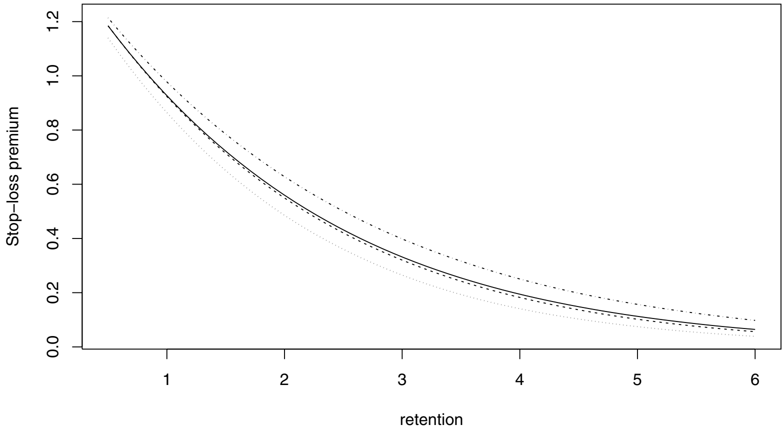

Fig. 1. Stop-loss premiums under exponential claims. The dot-dashed line (at the *top*) corresponds to the measure $Q^{(3)}$, the solid line is the minimal entropy measure, the dashed line is the minimal martingale measure, and the dotted line (*bottom*) corresponds to $Q^{(1)}$

with parameter δ. Recall that $g_{(\beta,\delta)}$ is the density for the gamma distribution with parameters (β, δ), see (4.6). Thus, in the example $G(dy) = g_{(1,\delta)}(y)dy$, so that $\mu_1 = 1/\delta$ and $\mu_2 = 2/\delta^2$. Using the various defining equations for the parameters appearing in the martingale measures presented in Table 6.2, we can characterize the Poisson intensities and the claim size distributions under the martingale measures in this example, see Table 2. This table shows that the claims are also exponentially distributed under the measures $Q^{(1)}$, $Q^{(3)}$ and $\overline{P}$, whereas the claim size distributions under $Q^{(2)}$ and $\widehat{P}$ are mixtures of certain exponential and gamma distributions with shape parameter 2. (Note in addition that the measure $Q^{(2)}$ is only defined if $\vartheta < 1$.)

We consider a numerical example, where we take $\lambda = \delta = T = 1$ and $\vartheta = 0.5$. Stop-loss premiums for retentions between 0.5 (which corresponds to $1/3$ of the premium, since $(1 + \vartheta)\mu_1\lambda = 1.5$) and 6 (4 times the premium) can be found in Fig. 1. (All numbers have been computed via simulation; an alternative idea would be to apply the so-called Panjer recursion, see Panjer 1981.) The figure illustrates the results in Proposition 6.4, in that the premiums are ordered for any retention levels such that indeed $F_{Q^{(1)}} \preceq_c F_{\widehat{P}} \preceq_c F_{\overline{P}} \preceq_c F_{Q^{(3)}}$. In particular, the figure confirms that premiums computed under the minimal martingale measure are smaller than the ones computed under the minimal entropy martingale measure. However, the difference between the premiums under these two measures is relatively small. For comparison, we have listed some of the premiums in Table 3, which also allows for some comments on the relation between the measures $Q^{(2)}$ and $\overline{P}$. For retention levels below 22, the measure $Q^{(2)}$ leads to higher prices than $\overline{P}$, which could seem to indicate that $F_{\overline{P}} \preceq_c F_{Q^{(2)}}$ in our example. However, for very high retention levels (above 22), $\overline{P}$ seems to lead to the highest prices. This is partly explained by the fact that $\overline{\mu}_1 \leq \mu^{Q^{(2)}}$, whereas the density of the claim size distribution for $\overline{P}$ is above the corresponding density for $Q^{(2)}$ for sufficiently large values of y. Thus, $\overline{P}$ and $Q^{(2)}$ are not ordered (in the convex order).

Table 3. Stop-loss premiums under the various martingale measures in the case of exponentially distributed claims

Measure/ retention	6	10	14	18	20	22	24
$Q^{(1)}$	0.039	0.0025	$1.3{\cdot}10^{-4}$	$6.4{\cdot}10^{-6}$	$1.4{\cdot}10^{-6}$	$3.5{\cdot}10^{-7}$	$1.0{\cdot}10^{-7}$
$\widehat{P}$	0.056	0.0047	$3.4{\cdot}10^{-4}$	$2.2{\cdot}10^{-5}$	$5.1{\cdot}10^{-6}$	$1.1{\cdot}10^{-6}$	$2.1{\cdot}10^{-7}$
$\overline{P}$	0.064	0.0063	$5.5{\cdot}10^{-4}$	$4.4{\cdot}10^{-5}$	$1.2{\cdot}10^{-5}$	$3.5{\cdot}10^{-6}$	$9.6{\cdot}10^{-7}$
$Q^{(2)}$	0.073	0.0072	$6.2{\cdot}10^{-4}$	$4.9{\cdot}10^{-5}$	$1.3{\cdot}10^{-5}$	$3.4{\cdot}10^{-6}$	$7.1{\cdot}10^{-7}$
$Q^{(3)}$	0.099	0.0140	$1.8{\cdot}10^{-3}$	$2.3{\cdot}10^{-4}$	$8.1{\cdot}10^{-5}$	$2.7{\cdot}10^{-5}$	$9.4{\cdot}10^{-6}$

6.4 Proof of Theorem 6.1

Assume for simplicity that $T = 1$. Consider two measures Q and $\widetilde{Q}$ with decompositions (5.1). The term U_1 is distributed as a standard compound Poisson variable under Q and $\widetilde{Q}$. Since X is a martingale under Q and $\widetilde{Q}$, the expected values of U_1 under the two martingale measures coincide. Thus, it is sufficient to check that the two compound Poisson distributions are ordered under the convex order. However, according to Condition 1, $\lambda^{\widetilde{Q}} \geq \lambda^{Q}$, so that we cannot use the results reviewed in Sect. 6.1 directly; the problem is that the claim arrival Poisson intensity under $\widetilde{Q}$ exceeds the Q-intensity, and we want to show that $F_{\widetilde{Q}} \preceq_c F_Q$.

In order to prepare for an application of the results on convex ordering, we first apply standard results for compound Poisson variables. The distribution of the compound Poisson variable under Q is identical to the distribution of another compound Poisson variable $\sum_{j=1}^{N_1'} Y_j'$, where $N_1' \sim$Poisson$(\lambda^{\widetilde{Q}})$, and where $Y_1', Y_2', \ldots$ are i.i.d., independent of N_1', with distribution G' on $[0,\infty) = \{0\} \cup (0,\infty)$ given by

$$\begin{aligned} G'(dy) &= \frac{\lambda^Q}{\lambda^{\widetilde{Q}}} G^Q(dy) + \left(1 - \frac{\lambda^Q}{\lambda^{\widetilde{Q}}}\right) \varepsilon_0(dy) \\ &= \frac{\lambda^Q}{\lambda^{\widetilde{Q}}} \frac{\lambda}{\lambda^Q} (1 + \phi^Q(y)) G(dy) + \left(1 - \frac{\lambda^Q}{\lambda^{\widetilde{Q}}}\right) \varepsilon_0(dy), \end{aligned} \tag{6.3}$$

where $\varepsilon_0(y)$ is the Dirac-measure at 0. For simplicity, one can take (here and in the following) all random variables equipped with a prime $'$ to be defined on a separate probability space $(\Omega', \mathcal{F}', P')$. Thus, we have increased the Q-Poisson intensity from λ^Q to $\lambda^{\widetilde{Q}}$ by the factor $\frac{\lambda^{\widetilde{Q}}}{\lambda^Q} \geq 1$ and replaced the claim size distribution G^Q by G', which is a mixture of the original Q-claim size distribution and the Dirac measure at 0, without affecting the distribution of the compound variable. (To see this, simply compute the characteristic functions of the two compound Poisson variables, and check that they are identical.) Thus, if we can show that $G^{\widetilde{Q}} \preceq_c G'$, then the assertion follows by using a result similar to (6.1). To see that $G^{\widetilde{Q}} \preceq_c G'$, we now apply the cut criterion to G', defined by (6.3), and to $G^{\widetilde{Q}}$ defined by $G^{\widetilde{Q}}(dy) = \frac{\lambda}{\lambda^{\widetilde{Q}}}(1 + \phi^{\widetilde{Q}}(y))G(dy)$. First note that the mean μ_1' in the distribution G' is identical to the mean $\mu_1^{\widetilde{Q}}$ in the distribution $G^{\widetilde{Q}}$, since the mean of the two compound Poisson variables coincide in the pure jump case. Denote by g' and $g^{\widetilde{Q}}$ the densities for the distributions G' and $G^{\widetilde{Q}}$ with respect to the convolution of G and the Dirac measure at 0. Then the difference between the two densities is

$$g'(y) - g^{\widetilde{Q}}(y) = \left(1 - \frac{\lambda^Q}{\lambda^{\widetilde{Q}}}\right) 1_{\{y=0\}} + \frac{\lambda}{\lambda^{\widetilde{Q}}} \left((1 + \phi^Q(y)) - (1 + \phi^{\widetilde{Q}}(y))\right) 1_{\{y \neq 0\}}.$$

Condition 2 of the above theorem guarantees the existence of $0 \leq y^1 \leq y^2 < \infty$ such that $\operatorname{sign}(g'(y) - g^{\widetilde{Q}}(y)) = +$, for $y \in [0, y^1) \cup (y^2, \infty)$, and $\operatorname{sign}(g'(y) - g^{\widetilde{Q}}(y)) = -$, for $y \in (y^1, y^2)$. Thus, according to the sufficient condition for the cut criterion, $G^{\widetilde{Q}} \preceq_c G'$, so that (6.1) gives that $F_{\widetilde{Q}} \preceq_c F_Q$ as claimed. In the case $\lambda^Q = \lambda^{\widetilde{Q}}$, no adjustment of the Poisson intensities is needed and the cut criterion follows by examining the original distribution functions G^Q and $G^{\widetilde{Q}}$ directly. □

Appendix: Extended framework

In this appendix we extend the results to the case of an inhomogeneous compound Poisson process and allow for a term driven by a Brownian motion. More precisely, we let

$$dX_t = \int_{(0,\infty)} y(\gamma(dt, dy) - \lambda_t\, G_t(dy)\, dt) + \sigma_t\, dW_t + (\lambda_t\, \mu_{1,t} - \kappa_t)\, dt, \quad \text{(A.1)}$$

where $X_0 = x_0$ and where $\gamma(dt, dy)$ is an integer valued random Poisson measure $\gamma(dt, dy)$ (i.e. a marked point process) with compensator $\nu(dt, dy) = G_t(dy)\, \lambda_t\, dt$. All processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and we fix a finite time horizon T. We let $\mathbb{F}$ be the P-augmentation of the natural filtration of X, we take $\mathcal{F}_T = \mathcal{F}$ and assume that $\mathcal{F}_0$ is trivial. The parameters λ_t, σ_t and κ_t are assumed to be deterministic functions of time t, and $G_t(\cdot)$ is a distribution function on $(0, \infty)$, which only depends on t. Thus, X has independent increments. The notation is now extended to allow for this dependence on time. For example, we denote by $\mu_{k,t}$ the k'th moment (under P) of claims occurring at t, and assume that $\kappa_t = (1 + \vartheta_t)\, \lambda_t\, \mu_{1,t} > \lambda_t\, \mu_{1,t}$, where $\vartheta_t > 0$. Finally, we assume that the distributions G_t have finite exponential moments, so that in particular $\mu_{k,t} < \infty$ for all $k > 0$. The process X can be written as $X = X_0 + M + A$, where M is a martingale, and where A is deterministic and of finite variation with

$$dM_t = \int_{(0,\infty)} y(\gamma(dt, dy) - \lambda_t\, G_t(dy)\, dt) + \sigma_t\, dW_t,$$
$$dA_t = (\lambda_t\, \mu_{1,t} - \kappa_t)\, dt = -\vartheta_t\, \lambda_t\, \mu_{1,t}\, dt.$$

The predictable quadratic variation process of M is $d\langle M \rangle_t = (\mu_{2,t}\lambda_t + \sigma_t^2)dt$, and we note that $dA_t = \alpha_t\, d\langle M \rangle_t$, with $\alpha_t = -\frac{\vartheta_t\, \lambda_t\, \mu_{1,t}}{\lambda_t\, \mu_{2,t} + \sigma_t^2}$.

Girsanov's Theorem

We recall Girsanov's Theorem for our situation, which is covered by Theorem III.3.24 of Jacod and Shiryaev (1987); see also Chan (1999). Consider a probability measure Q, with density process $Z_t = \mathrm{E}\left[\frac{dQ}{dP}\middle| \mathcal{F}_t\right]$, where Z is a P-martingale, $Z_0 = 1$, and

$$dZ_t = Z_{t-}\left(\psi_t dW_t + \int_{(0,\infty)} \phi_t(y)(\gamma(dt,dy) - \nu(dt,dy))\right),$$

with $\phi_t(y) > -1$ for predictable processes ψ and ϕ. Girsanov's Theorem states that W^Q defined by $dW_t^Q = dW_t - \psi_t dt$ is a Q-standard Brownian motion and that $\gamma(dt,dy) - \nu^Q(dt,dy)$ is a Q-martingale increment, where $\nu^Q(dt,dy) = (1+\phi_t(y))\nu(dt,dy)$. We get

$$\begin{aligned} dX_t = \sigma_t dW_t^Q + \psi_t\sigma_t dt + \int_{(0,\infty)} y(\gamma(dt,dy) - \nu^Q(dt,dy)) \\ + \int_{(0,\infty)} y\nu^Q(dt,dy) - (1+\vartheta_t)\mu_{1,t}\lambda_t dt, \end{aligned}$$

so that X is a (local) Q-martingale provided that (ψ,ϕ) satisfies the martingale equation

$$\psi_t\sigma_t + \lambda_t \int_{(0,\infty)} y(1+\phi_t(y))G_t(dy) - (1+\vartheta_t)\mu_{1,t}\lambda_t = 0, \tag{A.2}$$

for all t. If $\phi_t(y)$ is deterministic, i.e. a function of (t,y) only, we rewrite the Q-compensator as $\nu^Q(dt,dy) = \lambda_t^Q\, dt\, G_t^Q(dy)$, where $\lambda_t^Q = (1+\widetilde{\phi}_t)\lambda_t$, with $\widetilde{\phi}_t = \int_{(0,\infty)} \phi_t(y)G_t(dy)$, and

$$G_t^Q(dy) = \frac{1+\phi_t(y)}{1+\widetilde{\phi}_t}G_t(dy) = \frac{\lambda_t}{\lambda_t^Q}(1+\phi_t(y))G_t(dy). \tag{A.3}$$

In order for the process X to have independent increments under Q, it is necessary that the kernel ϕ is deterministic, see e.g., Jacod and Shiryaev (1987, Theorem II.4.15). We finally note that $\frac{dQ}{dP} = Z_T$ and that the density process Z may be written as

$$\begin{aligned} Z_t = \exp\left(\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right) \exp\left(\int_0^t \int_{(0,\infty)} \log(1+\phi_s(y))\gamma(dt,dy)\right) \\ \exp\left(-\int_0^t \int_{(0,\infty)} \phi_s(y)\nu(ds,dy)\right). \end{aligned}$$

The minimal martingale measure
Define a local martingale $\widehat{Z}$ via $d\widehat{Z}_t = -\widehat{Z}_t\, \alpha_t\, dM_t$ and $\widehat{Z}_0 = 1$. The minimal martingale measure $\widehat{P}$ is defined by $\frac{d\widehat{P}}{dP} = \widehat{Z}_T$, if $\widehat{Z}_T$ is strictly positive and $\mathrm{E}[\widehat{Z}_T] = 1$. Since $\alpha_t < 0$,

$$\begin{aligned} \widehat{Z}_t = \exp\left(-\int_0^t \alpha_s\, \sigma_s dW_s - \frac{1}{2}\int_0^t \alpha_s^2\, \sigma_s^2\, ds\right) \\ \exp\left(\int_0^t \alpha_s\, \lambda_s\, \mu_{1,s}\, ds\right) \exp\left(\int_0^t \int_{(0,\infty)} \log(1-\alpha_s\, y)\gamma(ds,dy)\right). \end{aligned} \tag{A.4}$$

By Girsanov's Theorem, the $\widehat{P}$-compensator of $\gamma(ds, dy)$ is

$$\widehat{\nu}(dt, dy) = (1 - \alpha_t\, y)\nu(dt, dy) = \frac{1 - \alpha_t\, y}{1 - \alpha_t\, \mu_{1,t}} G_t(dy)(1 - \alpha_t\, \mu_{1,t})\lambda_t\, dt. \tag{A.5}$$

Under $\widehat{P}$, $\widehat{\lambda}_t := (1 - \alpha_t\, \mu_{1,t})\lambda_t$, and $\widehat{G}_t(dy) := \frac{1-\alpha_t\, y}{1-\alpha_t\, \mu_{1,t}} G_t(dy)$.

The minimal entropy martingale measure

The minimal entropy martingale measure is of the form $\frac{d\overline{P}}{dP} = C \exp\left(\int_0^T \eta_s\, dX_s\right)$, for a constant C and some integrable process η. For a *deterministic* process η, this means that

$$\frac{d\overline{P}^\eta}{dP} = C\mathcal{E}\left(\int \eta\sigma dW\right)_T \exp\left(\int_0^T \int_{(0,\infty)} \log e^{\eta_t y}\gamma(dt, dy)\right). \tag{A.6}$$

We see that the $\overline{P}^\eta$-compensator of $\gamma(dt, dy)$ is $\overline{\nu}(dt, dy) = e^{\eta_t y}\nu(dt, dy) = \overline{G}_t(dy)\overline{\lambda}_t dt$, where $\overline{G}_t(dy) = \frac{e^{\eta_t y}}{\psi_t(\eta_t)} G_t(dy)$, and $\overline{\lambda}_t = \psi_t(\eta_t)\lambda_t$, and where $\psi_t(\eta_t) = \int_{(0,\infty)} e^{\eta_t y} G_t(dy)$. This leads to the martingale equation

$$\eta_t \sigma_t^2 + \lambda_t \int_{(0,\infty)} y e^{\eta_t y} G_t(dy) - (1 + \vartheta_t)\lambda_t \mu_{1,t} = 0. \tag{A.7}$$

Comparisons of optimal martingale measures

Let Q be some martingale measure for X and let

$$X_T = X_0 + \int_0^T \int_{(0,\infty)} y(\gamma(dt, dy) - (1+\phi_t^Q(y))\nu(dt, dy)) + \int_0^T \sigma_t\, dW_t^Q, \tag{A.8}$$

where W^Q is a standard Brownian motion under Q, and where $(1+\phi_t^Q(y))\nu(dt, dy)$ is the Q-compensator of $\gamma(dt, dy)$. According to (A.5), $\phi_t^{\widehat{P}}(y) = -\alpha_t y$ for the minimal martingale measure. For the minimal entropy martingale measure, $\phi_t^{\overline{P}}(y) = \exp(\eta_t y) - 1$, where η_t solves the Eq. (A.7). Consider now another martingale measure $\tilde{Q}$ and the corresponding decomposition. The following result is taken from Theorem 6.1 of Henderson and Hobson (2003), who prove this within an *exponential* jump-diffusion model via coupling and Jensen's inequality for conditional expectations.

Proposition A.1 (Henderson and Hobson 2003) *Let* $H = \Phi(X_T)$ *for some convex function* Φ *and consider martingale measures* $(Q, \widetilde{Q})$ *with* $(\phi^Q, \phi^{\widetilde{Q}})$ *deterministic. If* $\phi_t^Q(y) \geq \phi_t^{\widetilde{Q}}(y)$ *for all* (t, y)*, then* $\mathrm{E}_Q[\Phi(X_T)] \geq \mathrm{E}_{\widetilde{Q}}[\Phi(X_T)]$.

In our model, the kernels for the minimal martingale measure and the minimal entropy martingale measures do not admit the uniform ordering needed in the proposition. To see this, first note that $0 < \eta_t < -\alpha_t$. (This can be shown as in the homogeneous case.) Let $v_t(y) = e^{\eta_t y} - 1 + \alpha_t y = \phi_t^{\overline{P}}(y) - \phi_t^{\widehat{P}}(y)$, which is the difference between the kernels under the two optimal martingale measures. It follows that $v_t(y)$ is strictly positive for y sufficiently big, whereas $v_t(0) = 0$

and $v_t'(0) = \eta_t + \alpha_t < 0$. Thus, $v_t(y)$ is negative for small y, which shows that $\phi_t^{\overline{P}}(y) < \phi_t^{\widehat{P}}(y)$ for y small and $\phi_t^{\overline{P}}(y) > \phi_t^{\widehat{P}}(y)$ for y sufficiently big. In particular, we cannot apply Proposition A.1 for the two optimal measures.

Ordering of expected values in random Poisson parts
By comparing the martingale equations for $\widehat{P}$ and $\overline{P}$, we get that

$$-\alpha_t\sigma_t^2 + \lambda_t \int_{(0,\infty)} y(1-\alpha_t y)G_t(dy) = \eta_t\sigma_t^2 + \lambda_t \int_{(0,\infty)} ye^{\eta_t y}G_t(dy),$$

which implies that

$$\begin{aligned}\widehat{\lambda}_t\widehat{\mu}_{1,t} = \lambda_t \int_{(0,\infty)} y(1-\alpha_t y)G_t(dy) &= (\eta_t+\alpha_t)\sigma_t^2 + \lambda_t \int_{(0,\infty)} ye^{\eta_t y}G_t(dy)\\ &\le \lambda_t \int_{(0,\infty)} ye^{\eta_t y}G_t(dy) = \overline{\lambda}_t\overline{\mu}_{1,t}.\end{aligned}$$

Thus, the expected value under the minimal entropy martingale measure of the random Poisson part exceeds the expected value under the minimal martingale measure.

Ordering of expected values in claim size distributions
In the current model we cannot conclude that $\widehat{\lambda}_t \ge \overline{\lambda}_t$ via calculations similar to the ones used in the homogeneous pure jump case. However, similar arguments show that

$$\widehat{\mu}_{1,t} = \frac{\int y(1-\alpha_t y)G_t(dy)}{\int (1-\alpha_t y)G_t(dy)} \le \frac{\int ye^{\eta_t y}G_t(dy)}{\int e^{\eta_t y}G_t(dy)} = \overline{\mu}_{1,t}. \tag{A.9}$$

The inequality in (A.9) can be established by using Taylor expansions of the exponential functions appearing on the right of the inequality. Thus, it is sufficient to show that

$$(\mu_{1,t} - \alpha_t\mu_{2,t}) \sum_{m=0}^{\infty} \mu_{m,t}\frac{\eta_t^m}{m!} \le (1-\alpha_t\mu_{1,t}) \sum_{m=0}^{\infty} \mu_{m+1,t}\frac{\eta_t^m}{m!}.$$

Stochastic orders and optimal martingale measures
The extension of Theorem 6.1, which can be used to show that $\widehat{\mathrm{E}}[\Phi(X_T)] \le \overline{\mathrm{E}}[\Phi(X_T)]$ for any convex function Φ, now takes the form:

Theorem A.2 *Consider two equivalent martingale measures Q and $\widetilde{Q}$ with deterministic kernels (ψ_t^Q, ϕ_t^Q) and $(\psi_t^{\widetilde{Q}}, \phi_t^{\widetilde{Q}})$ and parameters (λ_t^Q, G_t^Q) and $(\lambda_t^{\widetilde{Q}}, G_t^{\widetilde{Q}})$. Let*

$$v_t^*(y) = \lambda_t^{\widetilde{Q}}\mu_{1,t}^{\widetilde{Q}}(1+\phi_t^Q(y)) - \lambda_t^Q\mu_{1,t}^Q(1+\phi_t^{\widetilde{Q}}(y)). \tag{A.10}$$

Assume that for all $t \in [0,T]$:

1. $\mu_{1,t}^Q \ge \mu_{1,t}^{\widetilde{Q}}$ and $\lambda_t^Q\mu_{1,t}^Q \ge \lambda_t^{\widetilde{Q}}\mu_{1,t}^{\widetilde{Q}}$.
2. There exist constants $0 \le y_t^1 \le y_t^2 < \infty$, such that $v_t^*(y) \ge 0$ for $y \in (0, y_t^1) \cup (y_t^2, \infty)$ and $v_t^*(y) \le 0$ for $y \in (y_t^1, y_t^2)$.

Then $F_{\widetilde{Q}} \preceq_c F_Q$, i.e. for any convex function Φ, $\mathrm{E}_{\widetilde{Q}}[\Phi(X_T)] \le \mathrm{E}_Q[\Phi(X_T)]$.

Proof As in the homogeneous case, the result is proved by examining the representation formulas for X_T under the two measures. However, in the present situation, where parameters are time-dependent, we cannot immediately apply the results on the closedness of the convex order under random summation. We therefore use a result of Norberg (1993) which identifies the jump part $\int_0^T \int y\gamma(dt, dy)$ with a standard compound Poisson variable. Consider first the jump part under $\widetilde{Q}$. By Theorem 1 and Corollary 1 of Norberg (1993),

$$\int_0^T \int_{(0,\infty)} y\gamma(dt, dy) \stackrel{\mathcal{D}}{=} \sum_{j=1}^{\widetilde{N}'_T} \widetilde{Y}'_j, \tag{A.11}$$

where the standard compound Poisson variable has parameters $\widetilde{\Lambda}_T = \int_0^T \lambda_t^{\widetilde{Q}} dt$ and

$$\widetilde{G}^*(dy) = \frac{\int_0^T \lambda_t^{\widetilde{Q}} G_t^{\widetilde{Q}}(dy)dt}{\int_0^T \lambda_t^{\widetilde{Q}} dt}. \tag{A.12}$$

The result says that, when considering the total claim amount during $[0, T]$, we can equivalently view claims as taken from the same distribution $\widetilde{G}^*$. More precisely, $\widetilde{G}^*$ is a mixture of the distributions $\{G_\theta^{\widetilde{Q}} | \theta \in [0, T]\}$ with a mixing distribution, which has density $h(\theta) = \lambda_\theta^{\widetilde{Q}} / \widetilde{\Lambda}_T$ on $[0, T]$. The proof consists in deriving a similar characterization of the jump part under Q and using the closedness of the convex order under mixtures.

We now turn to the distribution of the jump part under Q. Since the expected values of the jump parts under the two measures might differ, see Condition 1, we first decompose the Q-jump part into two (inhomogeneous) Poisson random measures. Let $\lambda'_t = (\lambda_t^{\widetilde{Q}} \mu_{1,t}^{\widetilde{Q}})/\mu_{1,t}^Q$ and $\lambda''_t = \lambda_t^Q - \lambda'_t$. The second part of Condition 1 ensures that $\lambda'_t \le \lambda_t^Q$, such that $\lambda''_t \ge 0$, and the first part shows that $\lambda'_t \le \lambda_t^{\widetilde{Q}}$. Let γ' and γ'' be independent random Poisson measures with compensators $\nu'(dt, dy) = \lambda'_t G_t^Q(dy)dt$ and $\nu''(dt, dy) = \lambda''_t G_t^Q(dy)dt$, respectively. Then it follows that under Q:

$$\int_0^T \int_{(0,\infty)} y(\gamma - \nu^Q) \stackrel{\mathcal{D}}{=} \int_0^T \int_{(0,\infty)} y(\gamma' - \nu') + \int_0^T \int_{(0,\infty)} y(\gamma'' - \nu'').$$

The compensated jump parts have mean zero. In particular, Jensen's inequality implies that the last term is larger in the convex order than 0 (its mean); consequently, this term can be ignored. We derive an ordering result for the first jump part and (A.11). Since

$$\int_0^T \int_{(0,\infty)} y\nu'(dt, dy) = \lambda'_t \mu_{1,t}^Q = \lambda_t^{\widetilde{Q}} \mu_{1,t}^{\widetilde{Q}} = \int_0^T \int_{(0,\infty)} y\nu^{\widetilde{Q}}(dt, dy),$$

for all t, we can focus on the jumps only. Next, we identify γ' with another random Poisson measure γ''' with compensator

$$\nu'''(dt, dy) = \lambda_t^{\widetilde{Q}} dt \left(\frac{\lambda'_t}{\lambda_t^{\widetilde{Q}}} G_t^Q(dy) + \left(1 - \frac{\lambda'_t}{\lambda_t^{\widetilde{Q}}}\right) \varepsilon_0(dy) \right) =: \lambda_t^{\widetilde{Q}} dt G_t'''(dy).$$

Thus, we can increase the intensity of claims from λ'_t to $\lambda_t^{\widetilde{Q}}$ and replace the distribution G_t^Q by a mixture of this distribution and the Dirac measure at 0. This change does not affect the distribution of the total claim amount, so that

$$\int_0^T \int_{(0,\infty)} y\gamma'(dt,dy) \overset{\mathcal{D}}{=} \int_0^T \int_{(0,\infty)} y\gamma'''(dt,dy) \overset{\mathcal{D}}{=} \sum_{j=1}^{N'_T} Y'_j, \tag{A.13}$$

where the last term is a standard compound Poisson variable with parameters $\widetilde{\Lambda}_T$ and

$$G^*(dy) = \frac{\int_0^T \lambda_t^{\widetilde{Q}} G_t'''(dy)dt}{\int_0^T \lambda_t^{\widetilde{Q}} dt}.$$

The second equality in distribution in (A.13) follows from Norberg (1993). Condition 2 of the theorem ensures that the sufficient condition for the cut criterion can be applied on $G_t^{\widetilde{Q}}$ and G_t'''. To see this, note that the difference between the densities for G_t''' and $G_t^{\widetilde{Q}}$ with respect to the convolution of G and the Dirac measure at 0 is

$$\begin{aligned} g_t'''(y) - g_t^{\widetilde{Q}}(y) &= \frac{\lambda'_t}{\lambda^{\widetilde{Q}_t}} \frac{\lambda_t}{\lambda_t^Q}(1+\phi_t^Q(y)) - \frac{\lambda_t}{\lambda_t^{\widetilde{Q}}}(1+\phi_t^{\widetilde{Q}}(y)) \\ &= \frac{\lambda_t}{\lambda_t^{\widetilde{Q}} \lambda_t^Q \mu_t^Q} \left(\lambda_t^{\widetilde{Q}} \mu_{1,t}^{\widetilde{Q}} (1+\phi_t^Q(y)) - \lambda_t^Q \mu_{1,t}^Q (1+\phi_t^{\widetilde{Q}}(y)) \right), \end{aligned}$$

for $y \in (0,\infty)$, where we have used the definition of λ'_t in the second equality. Similarly, $g_t'''(0) - g_t^{\widetilde{Q}}(0) = 1 - \lambda'_t/\lambda_t^{\widetilde{Q}} = 1 - \mu_{1,t}^{\widetilde{Q}}/\mu_{1,t}^Q$. The second condition in the theorem guarantees that we can apply the cut criterion to obtain that $G_t^{\widetilde{Q}} \preceq_c G_t'''$ for each t, and the closedness of the convex order under mixtures implies that $\widetilde{G}^* \preceq_c G^*$. Finally, the theorem follows by using that the convex order is closed under convolution and random summation. □

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