Discussion of *Distribution Based Pricing Formulas Are Not Arbitrage-Free*

David Ruhm is to be congratulated on an interesting and attention-grabbing paper. The apparent paradox of the central example creates a focus that will provide actuaries with insight into arbitrage-free pricing methods and their application to insurance pricing. We would like to comment on three issues:

1. Arbitrage-free pricing in the Black-Scholes model
2. Additive pricing formulas in insurance
3. Does the lack of continuous trading in insurance allow pricers to ignore arbitrage?

Arbitrage-free pricing in the Black-Scholes model

Actuaries studying the Black-Scholes model tend to struggle to relate it to the idea of risk-loading in insurance pricing. In Black-Scholes, everything is assumed to grow at the risk-free rate, and then you just take expected value as the price! Where is the risk load?

David's example shows that there is indeed risk load in this pricing, both at the speculative and risk-management sides of the distribution. That is a useful insight for actuaries, but where does this risk load come from? Perhaps another paradox is that in fact it comes from assuming the risk-free rate.

A risky security should have an expected return greater than the risk-free rate. So assuming it's mean return is the risk-free rate gives it a higher price today, because it will have less discount from its ending price. Thus discounting at the risk-free rate is not a neutral assumption. It builds in a valuation that may be appropriate, as in Black-Scholes, but may in other cases overstate the value of a position. This would be the case for a risky underlying asset or liability which is valued based on its ending distribution, as its value today would be less than the discounted value under the risk-free rate.

Additive pricing formulas in insurance

David's example shows that you cannot guarantee arbitrage-free pricing for derivative securities by taking transforms of the probability distributions of their outcomes. But he does not suggest that this finding invalidates Black-Scholes. He allows that you can get arbitrage-free pricing of derivative securities by taking appropriate transformations of the probability distributions of the underlying securities.

What does this mean for insurance pricing? We would like to argue that the underlying probability distributions are the ground-up frequency and severity distributions. Individual insurance and deals can be considered to be derivatives of these fundamental processes. Thus pricing by taking the mean of a transformation of the ending loss distribution for a deal will not necessarily give arbitrage-free pricing. But taking appropriate transforms of the underlying frequency and severity distribution will. What is appropriate? This will be discussed further below.

This conclusion is not entirely new. For instance, Wang PCAS 199? showed that pricing by taking means under transforms of aggregate losses for a contract will result in sub-additive pricing. Venter PCAS 199? commented that transforming the frequency and severity distributions separately would give additive pricing. But it is David's example that gives perspective to this discussion: transforming probabilities of deal outcomes in general does not lead to arbitrage-free pricing.

Additive pricing is close to arbitrage-free pricing in insurance. If the market prices a portfolio of risks at less than the sum of the prices of the individual risks, then an insurer can get an arbitrage profit by reinsuring its business 100%. Or if dividing an excess-of-loss program into three layers gives a lower price than making it two layers, a reinsurer could write it as two layers and retro-cede as three, and make a risk-free profit.

Additive pricing is a necessary condition for arbitrage-free pricing in insurance. Venter ASTIN 1991 presented a line of reasoning that in this context shows that additive pricing is equivalent to pricing with means of transformed frequency and severity distributions. This includes covariance pricing, such as CAPM, which can be expressed as a form of transformed probabilities. But is additivity sufficient for arbitrage-free pricing in insurance? This is part of what is addressed in the next section.

Does the lack of continuous trading in insurance allow pricers to ignore arbitrage?

David may let actuaries off the hook too easily on this issue. As the examples above show, just being able to cede contracts through reinsurance, even without continuous trading, can provide arbitrage opportunities. So at least additivity is required in insurance markets. A company's pricing preferences may not be additive, however. Initial models of a company as a publicly traded entity might suggest they should be, but those models are coming under increasing scrutiny. Thus a wider range of pricing formulas could be applied to generate company risk load targets, including probability transforms of individual deals, or even old standbys such as variance or standard deviation loadings.

Without continuous trading, mean pricing from transformed probabilities is enough to prevent arbitrage as long as the original and transformed distributions agree on which events have zero probability. Because of no continuous trading, the only events that have to be considered for arbitrage are the ending results. If there is a position that has some chance of a profit but no chance of a loss in the actual probabilities, then it will be in this same condition with the transformed probabilities. Thus its transformed mean, and therefore price, must be positive

Continuous trading and complete markets provide further constraints on which transformations will be free of arbitrage. For instance, with the Black-Scholes assumptions, transforms other than replacing the trend with the risk-free rate will not be arbitrage-free. Continuous trading strategies can be devised to make arbitrage profits against any other transform. However, it should be noted that no actual securities traded follow these assumptions, as prices of options do not agree with the Black-Scholes formulas with a fixed volatility for all options. They can be made to agree with the formula if you allow a different volatility for each option, but you could do that with any formula.

Another example is the one the Australian guys gave in Cancun. Shaun - do you have details?

This is not to imply that all continuously traded options have unique prices. In fact most, if not all, markets are incomplete, and these allow a wide variety of probability transforms to be arbitrage-free.

Conclusions

David's paper yields valuable insights into insurance and options pricing. It is especially useful in relating options pricing to risk loading, and also in showing that pricing based on transforms of the distributions of deal outcomes is not arbitrage-free. However, it should not be interpreted to imply that probability transforms in insurance never yield arbitrage-free pricing. As long as probability transforms are applied to the underlying frequency and severity distributions, and they agree with the actual probabilities on which events are impossible, they will provide arbitrage-free pricing of insurance deals. This does depend on markets not being complete with continuous trading, but even in incomplete markets, probability transforms are the only way to price without arbitrage.