DISTORTED PROBABILITY ≠ PREMIUM PRINCIPLE

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ABSTRACT

Distorted probability expected values produce premium pricing methods that differ from premium principles expressed with the exact same formulas.

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A number of authors have shown that arbitrage can be avoided by calculating premiums as the expected value of losses using distorted loss probability distributions. E.g., see Delbaen and Haezendonck (1989) or Venter (1991). However, a somewhat subtle issue arises when converting the distorted probability mean to a premium principle, and this can result in premium calculation with arbitrage possibilities.

For instance, Delbaen and Haezendonck consider the distorted probability density $f^*(x) = f(x)[1+b(x-E(X))]$. This has mean $E^*(X)= E(X) + bVar(X)$. This is arbitrage-free. However the variance premium principle $\pi(X) = E(X) + bVar(X)$ is not. To see this, suppose an insurer writes a policy priced according to the variance principle and cedes it to two reinsurers who each take half. Each reinsurer's loss variance is ¼ of that of the original risk's, so by the variance principle they would each get $0.5E(X) + 0.25bVar(X)$. Thus the original insurer would retain no risk but half of the loading.

On the other hand, if the reinsurers price according to the distorted probability mean, they would each take $E^*(0.5X) = 0.5E(X) + 0.5bVar(X)$, so there would be no arbitrage.

The difference arises because the premium principle is a formula that applies to each risk being priced, whereas the distorted probability method changes the probability measure once, and then applies those probabilities to each random variable under consideration. In order to get the results of the variance principle, you would have to establish a new measure $f^{**}(x) = f(x)[1+0.5b(x-E(X))]$ to apply to $X/2$, but keep the original f^*

for pricing X. Thus different probability measures would be used for the same events in the two calculations. Just using distorted probabilities does not guarantee arbitrage-free pricing unless you maintain consistent probabilities throughout the calculation.

This issue arises with a number of distorted probability measures that have been proposed historically. For instance, Buhlmann (1980) considers the Esscher transform $f^*(x) = e^{hx}f(x)/E(e^{hX})$. This leads to a price of $E^*(X) = E(Xe^{hX})/E(e^{hX})$. This transform provides arbitrage-free pricing, but again the associated premium principle $\pi(X) = E(Xe^{hX})/E(e^{hX})$ does not, as it is not additive for correlated risks. The same is true for the PH-transform $F^*(X) = 1 - (1 - F(X))^{1/\rho}$ suggested by Wang (1995).

Another probability transformation is suggested by the results of Gerber and Shiu (1994), who apply the Esscher transform to ln X for a few distributions. They show that this recovers some traditional financial pricing formulas. For instance, applied to lognormally distributed security prices, it gives the Black-Scholes option pricing formula as the adjusted expected value. The log-Esscher transform can be calculated using the moment transform $f^*(x) = x^r f(x)/E(X^r)$. The adjusted mean is $E^*(X) = E(X^{r+1})/E(X^r)$. Again this is arbitrage-free as an expected probability, but not as a premium principle. The calculation of these probabilities for layer pricing, etc. is not too difficult for many popular distributions, because the lognormal, transformed beta, transformed gamma, and inverse transformed gamma distributions all keep their same form under the moment or log-Esscher transformation.

Delbaen and Haezendonck consider transformations applied to compound Poisson processes, and recommend adjusting the Poisson mean and then applying a probability distortion such as those considered above to the severity distribution. Even when the frequency distribution is not Poisson, applying separate distortions to frequency and severity probabilities would seem preferable. This would be necessary for calculating reinsurance premiums for a variety of covers, for example.

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