The state-space model in the Verall and Zehnwirth references provides a formal statistical treatment of the types of instability in a data triangle. This model can be used to help analyze whether to use all the data, or to adopt some form of weighted average that de-emphasizes older data.

### *State-Space Model*

The application of state-space model differs from the alternative emergence methods in that it addresses instability in emergence patterns. As such it could be used in conjunction with any of those methods. It incorporates weights that emphasize the more recent data. Rather than discarding older data, it uses decreasing weights for older observations that gradually de-emphasize them. The Zehnwirth and Verrall papers in the References discuss this model in some detail.

The model assumes that observations fluctuate around a mean that itself changes over time. The degree of random fluctuation is measured by variance around the mean, and the movement of the mean by its variance over time. The interplay of these two variances determines the weights to apply, as in credibility theory.

Thus the state-space model is a discipline for identifying a central factor at each point in time amid random fluctuations. This problem is analogous to finding the signal amid the noise in a static-filled broadcast. The general idea of this model is that there is a true factor at each point in time, but this changes over time. Moreover, the factor itself is never observed directly; only a random observation somewhere in the vicinity of the factor is observed.

The state-space model thus provides underlying assumptions about the process by which development changes over time. With such a model, estimation techniques that minimize prediction errors can be developed for the changing development case. This can result in estimators that are better than either using all data, or taking the average of the last few diagonals. This is fairly detailed mathematically, and so is included in Appendix 1.

## Appendix 1– State Space Model

The problem is to find a reasonable estimate for the factor at each point that recognizes changes in the factor but is not overly responsive to random fluctuations. How responsive turns out to be a function of two important variances: the variance of the movement of the factor over time, and the variance of the random fluctuation. A formal definition of the model follows in the case of 2nd to 3rd development.

$\beta_i$=2nd to 3rd factor for ith accident year

$y_i$=3rd report losses for ith accident year

$x_i$=2nd report losses for ith accident year

$y_i=x_i\beta_i+\varepsilon_i$. The error term $\varepsilon_i$ is assumed to have mean 0 and variance $\sigma_i^2$.

$\beta_i=\beta_{i-1}+\delta_i$. The fluctuation $\delta_i$ is assumed to have mean 0 and variance $\nu_i^2$, and to be independent of the $\varepsilon$'s.

What needs to be estimated primarily are the factors $\beta_i$. These will be applied when $y_{i-1}$ and $x_i$ have been observed. The following notation will be used for the quantities to be estimated.

$\hat{\beta}_i$= estimate of $\beta_i$ given $y_1, \ldots, y_{i-1}$ and $x_1, \ldots, x_{i-1}$.

$\hat{y}_i$= forecast of $y_i$ given $y_1, \ldots, y_{i-1}$ and $x_1, \ldots, x_i$.

$\Gamma_i$= Variance of $\hat{\beta}_i$ ($\Gamma_1$ is a given parameter, later ones are estimated.)

$H_i$= Variance of $\hat{y}_i$ (Includes process variance $\sigma_i$ and parameter variance $\Gamma_i$.)

The estimation in the state space model is an iterative process that proceeds one period at a time. The process is similar to credibility. From the signal processing background the estimation procedure is called the *Kalman filter* after its originator. The key is the so called *Kalman gain factor*, which in effect is a credibility weight to see how much the new observation should be allowed to change the previous estimate of the development factor. The iterative process at each stage has several steps as follows:

| | |
|---|---|
| $\hat{y}_i = x_i\hat{\beta}_i$ | Next observation forecast |
| $k_i = \Gamma_i x_i/(x_i^2\Gamma_i+\sigma_i^2)$ | Update Kalman gain (credibility) factor |
| $\Gamma_{i+1} = \Gamma_i+\nu_{i+1}-k_i x_i\Gamma_i$ | Update parameter variance - it increases because of fluctuation, but decreases due to more data |
| $\hat{\beta}_i = \hat{\beta}_i+k_i(y_i-\hat{y}_i)$ | Update parameter estimate, using Kalman gain factor |
| $H_i = x_i^2\Gamma_i+\sigma_i^2$ | Update forecast variance (and thus standard error) |

In order to begin this process, several starting parameters are needed that may have to be estimated. They are: $\Gamma_1$, $\hat{\beta}_1$, $\sigma_i$, $\nu_i$. Some simplifying assumptions can be made to ease this initial start-up.

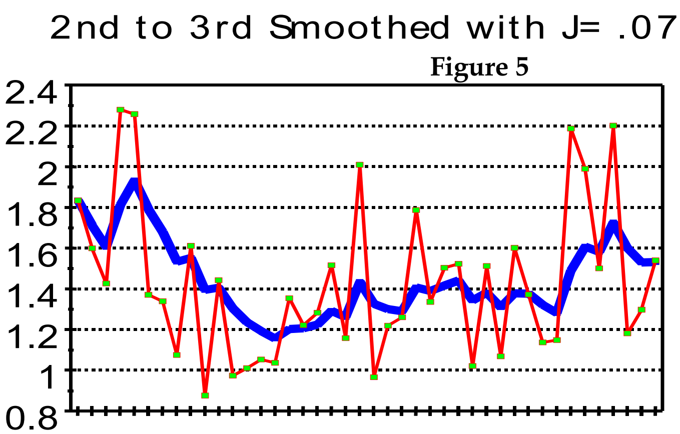


**Figure 5**

***Simplified State-Space Model***

Assumptions that greatly simplify this model are that $\nu_i=\nu$, and $\sigma_i=x_i\sigma$, for each period i. This leads to the result:

$$\hat{\beta}_{i+1}= \hat{\beta}_i+z_i(y_i/x_i-\hat{\beta}_i),$$

where

$$z_1= \Gamma_1/(\Gamma_1+\sigma^2) \text{ and}$$
$$z_{i+1}= 1/[1+1/(z_i+J)],$$

$J=\nu^2/\sigma^2$

This is the Gerber-Jones credibility model (e.g., see *Credibility Formulas with Geometric Weights* in the 1975 Society of Actuaries Transactions or the credibility chapter of the CAS textbook). In the limit the credibility is $z_\infty=½J[(1+4/J)^{½}-1]$. In this formula the development factor for an accident year is the previous year's factor plus a correction term, where the correction term is the credibility weight z times the prediction error from the last year. The formula can also be interpreted as an estimation of the development factor as a weighted average of the individual year development ratios, with the weights declining for more remote periods. To see this, let $w_i=z_iy_i/x_i$. Then successive substitution yields:

$$\hat{\beta}_{i+1}=w_i+(1-z_i)w_{i-1}+(1-z_i)(1-z_{i-1})w_{i-2}+..+\hat{\beta}_1\Pi(1-z_j),$$

which is a declining weighted average of the $y_j/x_j$'s. This still needs starting values. Assuming no prior knowledge, you could start with $z_1=1$ and $\hat{\beta}_1=y_1/x_1$. Then just J would be needed to do updates. Rather than estimating its component variances separately, J could be estimated by trying different J's to minimize the SSSSPE= $\Sigma_i[y_i/x_i-\hat{y}_i/x_i]^2$. SSSSPE stands for the sum of squares of single step prediction errors, which is a goodness-of-fit measure for varying parameter models. It quantifies how well the estimation actually performs in separating signal from noise. As an example, the fit with J=.07 is shown in Figure 5. That J yields a $z_\infty$ of 0.23. The fit appears to capture longer term changes without following random fluctuations too greatly. The values of $z_I$ and $\hat{\beta}_{i+1}$ that derive from the Gerber-Jones formula are shown in Exhibit 2. This starts with an initial z of unity, then updates with J=.07.

Now suppose, however, that there is a break in the data at the jth point. By setting $\nu_j$ large in the original Kalman updating formula, $\Gamma_j$ also becomes large, which leads to $k_j = 1/x_j$. This mean that the next estimate will be $\hat{\beta}_{j+1}=y_j/x_j$, that is the next observation gets full credibility. The updating

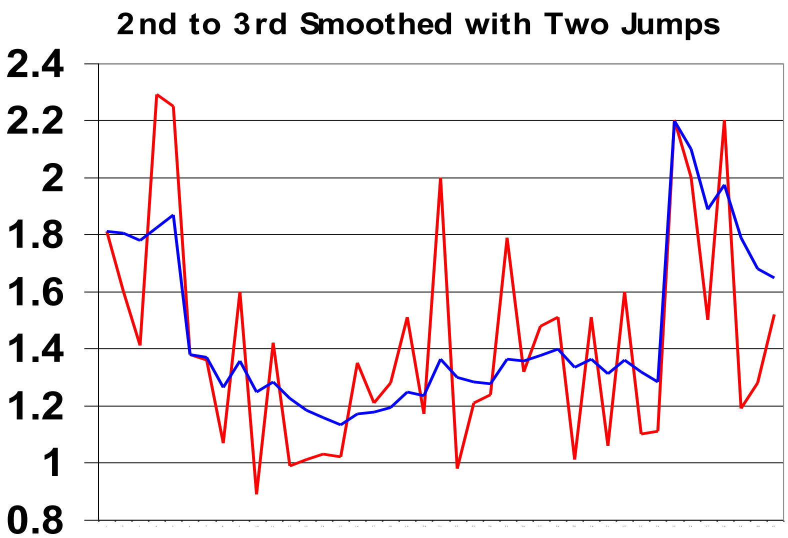


**Figure 6**

starts over from there. How large do these values have to be? Just large enough so that k=1/x to the number of decimal points in the calculation.

To illustrate this, Exhibit 2 also shows the updating calculation for the 2nd to 3rd factor above with a jump after the fifth point and another jump seven point from the end. These were chosen as points where there appears to be a change in level, not just a random fluctuation. For this example, $\nu$=0.003 except at the two points where it is large, $\sigma_i=\sigma$ =0.3, $x_I$=1, and k starts at 1. Here all the x's are unity, as the parameter is the estimate. This procedure gives J=1/30 except at the two jump points, which is less than half what it was in the first example. In effect, the process is assumed to be smoother except at the jumps. The effect is shown in Figure 6.

One way of comparing such models is the SSSSPE: sum of squares of single step prediction errors $(y_i-\hat{y}_i)^2$. This is 6.08 for the first smoothing and 5.42 with the jumps. This is a good measure of fit that does not require counting degrees of freedom. If too little smoothing is done, the fit may be better but the prediction errors could be worse, as the fit would be following random fluctuations. By taking other values of the parameters, more or less smoothing can be obtained. The SSSSPE can also be used to test other smoothing techniques, such as average the last 5 diagonals. For this series that method gives an SSSSPE of 6.25.

## Exhibit 2 – Simplified State-Space Example

| | J=.07 | | Two Jumps | | | |
|---|---|---|---|---|---|---|
| 2nd to 3rd | z | beta | Nu | Gamma | k | Beta |
| a | $b=1/[1+1/(.07+b_{-})]$ | $c=ab+c_{-}(1-b)$ | d | $e=d+e_{-}(1-f_{-})$ | $f=e/(e+0.3^2)$ | $g=af+g_{-}(1-f)$ |
| 1.81 | 1 | 1.81 | 0.003 | 0.003 | 1.000 | 1.81 |
| 1.60 | 0.517 | 1.70 | 0.003 | 0.003 | 0.032 | 1.80 |
| 1.41 | 0.370 | 1.59 | 0.003 | 0.006 | 0.062 | 1.78 |
| 2.29 | 0.305 | 1.81 | 0.003 | 0.009 | 0.087 | 1.82 |
| 2.25 | 0.273 | 1.93 | 0.003 | 0.011 | 0.107 | 1.87 |
| 1.38 | 0.255 | 1.79 | 1000000 | 1000000 | 1.000 | 1.38 |
| 1.36 | 0.246 | 1.68 | 0.003 | 0.093 | 0.508 | 1.37 |
| 1.07 | 0.240 | 1.54 | 0.003 | 0.049 | 0.351 | 1.26 |
| 1.60 | 0.237 | 1.55 | 0.003 | 0.035 | 0.278 | 1.36 |
| 0.89 | 0.235 | 1.40 | 0.003 | 0.028 | 0.237 | 1.25 |
| 1.42 | 0.233 | 1.40 | 0.003 | 0.024 | 0.213 | 1.28 |
| 0.99 | 0.233 | 1.31 | 0.003 | 0.022 | 0.198 | 1.23 |
| 1.01 | 0.232 | 1.24 | 0.003 | 0.021 | 0.188 | 1.19 |
| 1.03 | 0.232 | 1.19 | 0.003 | 0.020 | 0.181 | 1.16 |
| 1.02 | 0.232 | 1.15 | 0.003 | 0.019 | 0.176 | 1.13 |
| 1.35 | 0.232 | 1.20 | 0.003 | 0.019 | 0.173 | 1.17 |
| 1.21 | 0.232 | 1.20 | 0.003 | 0.019 | 0.171 | 1.18 |
| 1.28 | 0.232 | 1.22 | 0.003 | 0.018 | 0.170 | 1.19 |
| 1.51 | 0.232 | 1.29 | 0.003 | 0.018 | 0.169 | 1.25 |
| 1.17 | 0.232 | 1.26 | 0.003 | 0.018 | 0.168 | 1.23 |
| 2.00 | 0.232 | 1.43 | 0.003 | 0.018 | 0.168 | 1.36 |
| 0.98 | 0.232 | 1.33 | 0.003 | 0.018 | 0.167 | 1.30 |
| 1.21 | 0.232 | 1.30 | 0.003 | 0.018 | 0.167 | 1.28 |
| 1.24 | 0.232 | 1.29 | 0.003 | 0.018 | 0.167 | 1.28 |
| 1.79 | 0.232 | 1.40 | 0.003 | 0.018 | 0.167 | 1.36 |
| 1.32 | 0.232 | 1.38 | 0.003 | 0.018 | 0.167 | 1.36 |
| 1.48 | 0.232 | 1.41 | 0.003 | 0.018 | 0.167 | 1.38 |
| 1.51 | 0.232 | 1.43 | 0.003 | 0.018 | 0.167 | 1.40 |
| 1.01 | 0.232 | 1.33 | 0.003 | 0.018 | 0.167 | 1.33 |
| 1.51 | 0.232 | 1.37 | 0.003 | 0.018 | 0.167 | 1.36 |
| 1.06 | 0.232 | 1.30 | 0.003 | 0.018 | 0.167 | 1.31 |
| 1.6 | 0.232 | 1.37 | 0.003 | 0.018 | 0.167 | 1.36 |
| 1.10 | 0.232 | 1.31 | 0.003 | 0.018 | 0.167 | 1.32 |
| 1.11 | 0.232 | 1.26 | 0.003 | 0.018 | 0.167 | 1.28 |
| 2.20 | 0.232 | 1.48 | 1000000 | 1000000 | 1.000 | 2.20 |
| 2.00 | 0.232 | 1.60 | 0.003 | 0.093 | 0.508 | 2.10 |
| 1.50 | 0.232 | 1.58 | 0.003 | 0.049 | 0.351 | 1.89 |
| 2.20 | 0.232 | 1.72 | 0.003 | 0.035 | 0.278 | 1.97 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1.19 | 0.232 | 1.60 | 0.003 | 0.028 | 0.237 | 1.79 |
| 1.28 | 0.232 | 1.52 | 0.003 | 0.024 | 0.213 | 1.68 |
| 1.52 | 0.232 | 1.52 | 0.003 | 0.022 | 0.198 | 1.65 |