Coherent Measures, Co-Measures, and Capital

Expected Policyholder Deficit

$EPD = E[(X - a)S(a) \mid X>a]$

where X is losses, a is assets, and S(a) is 1 – F(a), i.e., S(a) is the probability that losses are greater than assets.

Define the risk leverage function g(X) as $S(a)(X - a)/(X - m)$, where m is E(X). Then:
$EPD = E[(X - m)g(X) \mid X>a]$ which is the form required to define co-measures. From this we can define co-EPD for any sub-portfolio X_i of X as:

$EPD_i = E[(X_i - m_i)g(X) \mid X>a] = E[S(a)(X - a)(X_i - m_i)/(X - m) \mid X>a]$

This is the expected contribution to default of the portfolio. It only looks at the cases of overall company default, i.e., X>a, and for those cases computes the total fraction of the default amount X – a due to the portfolio having adverse deviation. That is it takes the probability of default times the expected value over all the default scenarios of the default amount of the scenario times the ratio of adverse losses for the portfolio to adverse losses for the company for that scenario. Thus portfolio losses that come in at expected are not considered to be contributing to default and losses below expected are treated as reducing the default amount. The large defaults are probably going to occur when everything goes bad, so it is unlikely that any portfolio would have a negative co-EPD overall.

Allocating surplus in proportion to co-EPD would charge each portfolio in proportion to its contribution to overall company defaults. Various portfolios would have various probabilities and degrees of defaulting on their allocated surplus from this method, but that seems of less financial import than their impact on the overall default.

As with any co-measure, allocation can go down to sub-portfolios and be completely consistent and additive. For instance, you could allocate each profit center to the state level, then add up the profit centers for a state to get the overall state capital.

One problem with EPD is that the ruin probability is hard to calculate, as it is way out in the tail, where all probabilities are projections along uncertain curves of uncertain values. Instead of deficit due to default, you could alternatively measure deficit over some percentage of surplus, say 20%. For example, a company might want to carry enough surplus so that the 99th percentile loss only exhausts 20% of surplus. The deficits above this amount can be treated just the same way as deficits over total assets were treated for EPD. This would only change the definition of a in the EPD formula. For instance, define a = liabilities + 20% of surplus. Then the loss surplus deficit can be defined as:

$LSD = E[(X - a)S(a) \mid X>a]$

And co-LSD as:

$LSD_i = E[(X_i - m_i)g(X) \mid X>a] = E[S(a)(X - a)(X_i - m_i)/(X - m) \mid X>a]$, etc.

Tail Value at Risk

$TVAR_q = E[X \mid X>x_q]$ where $F(x_q) = q$. Note that if x_q = assets, then:

EPD = default probability * ($TVAR_q$ –assets).

TVAR is a coherent measure, which means, among other things, that for a fixed q the sum of the TVAR's of any collection of loss portfolios will be the same or greater than the TVAR of the combined portfolio[1]. The excess of the sum over the aggregate can be considered a diversification benefit. Thus if you allocate surplus by TVAR you will have to reduce the allocations for the diversification effect.

You can also express $TVAR_q = E[(X - m)g(x) \mid X>x_q]$ where $g(x) = X/(X - m)$, so co-TVAR is: $TVAR_{qi} = E[X(X_i - m_i)/(X - m) \mid X>x_q]$. Here we are assuming a point somewhat in the tail, with $x_q > m$ so X – m is positive in the expected value. The co-TVAR allocates to the ith portfolio the contribution of that portfolio to the overall TVAR . In the computation of the tail mean, each scenario with $X>x_q$ has the loss amount modified by the ratio of the portfolio adverse loss to the total adverse loss for that scenario.

[1] This is not always going to be the case for VAR or EPD