Co-Measures

Co-measures can be defined for any risk measure that can be expressed as a conditional expectation, which is most of them. Suppose a risk measure for risk X with mean m can be defined as:

$$R(X) = E[(X - am)g(x) \mid \text{condition}] \quad \text{for some value a and function g.}$$

Suppose further that X is the sum of n portfolios X_i each with mean m_i. Then the co-measure for X_i is:

$$\text{co-}R(X_i) = E[(X_i - am_i)g(x) \mid \text{condition}]$$

Here the condition is the same as in the definition of R, so it is a condition on X, not X_i. Since expectations are additive, the sum of the co-R's of the n X_i's is R(X).

Variance

As an example, take a=1 and g(X) =X – m, with any condition that is always fulfilled, like 0X=0. Then R(X) is the variance of X. Thus,

$$\text{co-}R(X_i) = E[(X_i - m_i)(X - m)], \text{ which is the covariance of } X_i \text{ with X.}$$

Value at Risk

Value at risk at probability level q can be defined as:

$$E(X \mid F(X)=q)$$

This is just the qth quantile of the distribution. Then the co-VaR is:

$$E(X_i \mid F(X)=q)$$

This would be the average value of portfolio i when total losses are at the qth quantile.

Tail Value at Risk

For probability level q, take a=0 and g(x) = 1, with condition F(X)>q. If q=99.9%, R is TVaR at the 1-in-1000 level. Then:

$$\text{co-TVaR}(X_i) = E[(X_i \mid F(X)>q)]$$

This is the mean loss for the ith unit in the case where total losses are over the q^{th} quantile.

Expected Policyholder Deficit

As another example, consider the expected policyholder deficit, or EPD. If X is all years'

losses unpaid, b is total assets, and $S(b)=1-F(b)$, then:

$$EPD = E[(X-b)S(b)\,|\,X>b]$$

This is the R(X) form with $a = 1$, $g(x) = S(b)(X-b)/(X-m)$ and condition $X>b$. With these, the co-measure is:

$$\text{Co-EPD}(X_i) = E[(X_i-m_i)g(X)\,|\;X>b]$$

$$= E[S(b)(X-b)(X_i-m_i)/(X-m)\,|\,X>b]$$

Each portfolio is allocated a fraction of the overall expected deficit given by the ratio of its losses above mean to the total losses above mean when there is a deficit.

Excess Tail Value at Risk

Define the measure excess tail value at risk by:

$$XTVaR_q = E[X-m\,|\,F(X)>q]\text{ , so}$$
$$\text{Co-}XTVaR_q = E[X_i-m_i\,|\,F(X)>q]$$

If capital is set by XTVaR, it would provide enough to cover losses above mean losses for the average of the years in which losses exceeded the qth quantile. The capital allocated by Co-XTVaR to a line would be the line's average losses above its mean losses in those same adverse years.

Discussion

Co-measures were introduced by Rodney Kreps as a way of allocating capital in an additive manner that is nonetheless consistent with the overall risk measure used to define total capital. They can be most easily thought of in terms of a scenario generator. Take the case where the total capital requirement is set to be the tail value at risk at the 1-in-1000 probability level. Then in generating scenarios, about 1 in 1000 would be above that level. The co-TVaR for each business unit would just be the average of its losses in those scenarios. This would be its contribution to the overall TVaR. This is a totally additive allocation. Business units could be combined or subdivided in any way and the co-TVaR's would add up. For instance, all the lines of business could be allocated capital by co-TVaR, then each of these allocated down to state level, and those added up to get the state-by-state capital levels for all lines combined. This could be done for peril or other business categories as well.

There should be some q for which XTVaR makes sense as a capital standard, as the mean loss should be already collected in premium. Using co-XTVaR for allocation would not charge capital to a unit for its mean losses. If by some chance the unit did not have losses above its mean in the average of the scenarios above the q^{th} quantile for the entire company, it would not be charged any capital. This makes sense if capital is indeed being held for the adverse outcomes.