Capital Allocation via Marginal Allocation of Risk Measures

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Abstract

Marginal allocation has some clear advantages in practice like business units not getting additional capital charges due to the actions of other business units. Virtually all of the risk measures used in practice can be formulated as means under transformed probability measures, which often facilitates the calculations. When allocation is done for risk pricing or risk-adjusted return, additional constraints on the risk measure are needed for consistency with pricing theory. Pricing is not the only application, however, as sometimes managers would like to know what businesses are driving capital requirements, even though the risk measure might not be appropriate for pricing. In the end, however, pricing by risk measures misses some actual financial issues and is at best an approximation.
Keywords. allocation, risk-adjusted return, equivalent martingale measure, risk measures, arbitrage.

1. INTRODUCTION

One criterion for allocation of business aggregate risk measures to business units is that if a unit makes a small proportional change in its volume, for instance by quota share reinsurance, the resulting change in the aggregate measure gets allocated back to that unit. If units are competing against each other for bonuses, resources, etc., this criterion is a prerequisite to having reasonable discussions among the units. It turns out that there are requirements on both the risk measure and the allocation method for this criterion to be met. Basically the risk measure has to be in currency units, which most are – except for variance which is measured in dollars squared. Also the allocation has to be proportional to the derivative of the aggregate risk measure with respect to the volume of the business unit. If the measure is in currency terms (i.e., homogeneous degree 1), then the derivatives of the units will add up to the risk measure and the criterion is met. For most risk measures the derivatives are simply expressed and well known. They are not always easy to calculate, however.

If the allocation is going to be used for pricing or for comparing risk-adjusted profitability across units, two more criteria are desirable:

1. If you make a small proportional increase in the volume of a unit with higher than average return, the return of the company will increase. This criterion is called suitability, and turns out to be equivalent to the previous criterion, which we will call marginal allocation.
2. The implied prices should be arbitrage-free. Not everyone agrees that this is necessary for in-

surance pricing, since not every possible deal actually exists in the insurance market (i.e., the market is not complete). But the insurance market is competitive enough to compete away arbitrage opportunities that might exist. In any case, this criterion requires that the risk measure can be defined as a specific type of probability transformation, known as an equivalent martingale transform. The main thing this requires is that the set of events with non-zero probability be the same in the original and transformed measure. This is violated for instance by TVaR, which is the mean under a transform that gives zero probability to all the events below the related VaR threshold.

A good deal of this is well known. We highlight here recent results from the literature and a few minor original findings. Section 2 discusses some misconceptions about equivalent martingale pricing. Section 3 introduces some lesser-known transforms, shows that the marginal allocation of standard deviation is an equivalent martingale, discusses criteria for selecting a transform in an incomplete market, and gives an example. Section 4 discusses allocation of VaR. Section 5 concludes.

2. DETAILS OF EQUIVALENT MARTINGALE PRICING

As noted above, an equivalent martingale measure agrees with the original measure on the set of events with zero probability. A good introduction to this area in an actuarial context is provided by Panjer (1998). One thing that this source makes clear (e.g., p. 180) is that the transformed probabilities have to be applied to the states of the world, not to the outcomes of particular deals, for the pricing to be arbitrage-free. Pricing every option by individual transforms of the distributions of the outcomes of each option will not in general produce arbitrage-free pricing. The transform has to be applied to the distribution of the outcomes of the original security, and every option priced by those probabilities, to get arbitrage-free prices.

Ruhm (2003) gives an entertaining example of this detail. Unfortunately the title of his paper, "Distribution-Based Pricing Formulas are not Arbitrage-Free" might lead the reader to believe he is proving a stronger conclusion, as his original discussant pointed out. Transforming outcomes is a pitfall that is easy to make, e.g., in layer pricing, and actuaries who try this often find out that this is the wrong way to apply transforms. Indeed, distribution-based pricing formulas are not arbitrage-free when you do them wrong.

Another detail about transformed-distribution pricing is that in complete markets, where every option on every security can be bought and sold, there is a unique transform that will produce arbitrage-free pricing in that market. However in incomplete markets, every equivalent martingale, when applied to market states, will be arbitrage-free. Thus if there are at least two complete markets each with their own transform, there can be no universal transform that will be arbitrage-free in every market. It is easy to believe that at least two such markets exist, so it is not likely that any transform would apply universally.

One popular transform, the Wang transform, defined below, does work in a geometric Brownian motion complete market as well as in all incomplete markets. Since geometric Brownian motion is a popular model, the Wang transform is as universal as a transform can get. Of course no known markets actually display geometric Brownian motion, so this is a bit hypothetical.

Pelsser (2008) gives another hypothetical example of a complete market, namely mean-reverting geometric Brownian motion. This is different from typical equities, in that if prices go up for a security, they would eventually come back down to the mean. He shows that such a market would have its own equivalent martingale transform. Thus we have two hypothetical markets with two different transforms, so we can at least imagine a world where there is no universal transform. Some standard models that work pretty well for equity prices, like the double exponential jump diffusion process, would probably provide other examples.

Pelsser concludes that "the Wang Transform cannot be a universal framework for pricing financial and insurance risks." This is of course a bit of an overstatement because it is based on hypothetical markets that probably do not exist. Yet it is easy to believe, not just for the Wang transform but for any transform. But he also says "We establish in this paper that, for general stochastic processes, the Wang Transform does not lead to a price which is consistent with the arbitrage-free price." This could lead a reader to believe that the Wang transform is not arbitrage-free in typical incomplete markets, which is not the case when it is applied correctly. Part of this arises from ambiguity surrounding the negation of the word "general," which can mean either "usually doesn't work" or "at least in one case doesn't work." It is not unusual to hear people say that the Wang transform has been shown not to be arbitrage-free, which is at least a misunderstanding of Pelsser's result.

3. EXAMPLES OF APPLICATION

To have some specific examples, first note that with parameter $k>0$, the Wang transform Q of a cumulative probability P may be computed as: $Q = \Phi[\Phi^{-1}(P) - k]$. Then $Q < P$, so more probability is in the tail after the transform.

Another transform, the exponential transform, is given by: $Q = [\exp(kP) - 1] / [\exp(k) - 1]$. Niederau and Zweifel (2009) show that this transform minimizes the information distance from the original measure under certain reasonable constraints. That sounds like a theoretically nice thing to happen, but the transform needs more testing against actual market prices.

If the units X_i add up to the business total X, the marginal allocation of std(X) to the unit X_i is given by the diversified standard deviation = $\text{cov}(X_i,X)/\text{std}(X) = \text{std}(X_i)\text{corr}(X_i,X)$. See for example Venter, Major, Kreps (2006). Thus the correlation of the unit with the business total is the factor to adjust the unit standard deviation for diversification. In some circumstances, this allocation is an equivalent martingale transform.

Following Venter (1991) we can show this by defining for pdf probability p the transform $q = p[1+k[E(X|X_i) - EX]/\text{std}(X)]$. Then it is not difficult to show that the transformed mean $E^*(X_i) = E(X_i) + k\ \text{cov}(X_i,X)/\text{std}(X)$. The main trick to this is to show that $E[X_iE(X|X_i)] = E[X_iX]$, which follows because $\int x_i f(x_i) \int x f(x|x_i) = \int x_i x f(x_i,x)$.

Since $E(X|X) = X$, the corresponding transform for X is $q = p[1+k[X - EX]/\text{std}(X)]$, so $E^*(X) = EX + k\ \text{std}(X)$. For this to be an equivalent transform, the ratio q/p has to stay positive and finite. This is at its minimum for loss X = 0, so for it to be positive requires that $1 > k/CV$, where CV is the ratio of standard deviation to mean. Thus the loading k must be less than CV for this to be an equivalent martingale transform. The standard deviation must exist for this to be practical as well.

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