

## Capital Allocation by Layer

This paper by Bodoff gives a new way to allocate VaR. It turns out to be a co-measure but not a marginal co-measure.

To formulate it, let  $f(x)$  be the probability of aggregate losses to the company of  $x$ , with survival function  $S(x) = 1 - F(x)$ . For a business unit B, let  $x_B$  be the average loss to unit B when total losses are  $x$ . For probability level  $\alpha$ , define:

$$A(x) = \int_0^{\min(x, F^{-1}(\alpha))} dt / S(t)$$

Then

$$E[A(x)] = \int_0^{\infty} f(x) \int_0^{\min(x, F^{-1}(\alpha))} dt / S(t) dx = \int_0^{F^{-1}(\alpha)} \int_t^{\infty} f(x) dx dt / S(t) = \int_0^{F^{-1}(\alpha)} S(t) dt / S(t) = F^{-1}(\alpha)$$

This gives a different way to express VaR as an expected value.

this seems strange to me – the math looks right but think of  $1/S(t)$  as the return time for a loss of  $t$  or greater. Then  $A(x)$  is the average return time for aggregate losses less than  $x$ . Its expected value would normally be less than VaR at  $\alpha$ , depending on units.

Let  $C(x) = A(x)/x$ . Then  $F^{-1}(\alpha) = E[XC(X)]$ . Allocate this to B by  $E[X_B C(X)]$ . This is a co-measure allocation. However it is not the marginal co-measure allocation.

The paper does not go into the properties of this allocation. The only property mentioned is “different from other methods.”

The idea that small losses should get a cost of capital seems good, as does the idea that larger losses should get more capital. I think where the reasoning breaks down is when this is considered an allocation of VaR.

When you get to the risk load section, basically before discounting a loss of  $x$  is charged its expected value plus  $f(x)rA(x)$ . Thus the premium is  $f(x)[x + rA(x)] = xf(x)[1 + rA(x)/x]$ . This is almost a transformed probability price – the only problem being that all the probabilities increase, so they don't integrate to 1. The total risk load is  $r\text{Var}$ .

Maybe an alternative way to look at it is just to allocate capital instead of VaR. So define  $A(x)$  with the limit of capital instead of  $F^{-1}(\alpha)$ .

Or allocate the risk load  $r^*\text{capital}$ . This maintains the principle of allocating by value of risk. Either way, you allocate capital by average return time. That does give more weight to larger losses and some weight to all losses. This might be setting capital to a spectral measure.