Distributions Underlying Power Function ILF's (Riebesell Revisited)

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Abstract:

A popular form for increased limits factors for heavy-tailed lines turns out to be closely related to a formula presented by Riebesell in his 1936 book introducing property-liability insurance mathematics. This note shows the underlying severity distributions that the factors imply.

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For heavy-tailed, data-sparse lines, especially in professional liability, it is becoming popular to use rates where the increased limited factor (ILF) at x times the selected base is given by:

$ILF(x*base) = x^a$, where $1 > a > 0.$

We show how this can be connected to a classical formula from the German actuary Riebesell (1936) and give examples of severity distributions that are consistent with it. Mack and Fackler (2003) discuss Riebesell's system and show how to generate severity distributions underlying it. We put their work into the x^a notation and provide some applications and extensions.

Riebesell starts with a pure premium of b_0 at base limits. His rule is that doubling the limit increases the premium by a factor of $1+z$. Thus increasing the limit by a factor of 2^r increases the premium by a factor of $(1+z)^r$. The power r on $1+z$ is the log based 2 of the increase in limits 2^r, which allows generalization to non-integer powers of 2. Denoting the log based 2 of x as $ld(x)$, the pure premium for a limit of $x*base$ is $b_0(1+z)^{ld(x)}$. Thus the ILF is $(1+z)^{ld(x)}$. This is the same as $x^{ld(1+z)}$, which can be seen by taking the log based 2 of both. Thus setting $a = ld(1+z)$ puts Riebesell's rule in the form used in excess professional liability lines.

In the notation of this note, Mack/Fackler show how to find a severity distribution consistent with starting values of a and k, where $k*base$ is the limited average severity at the base – denoted here as *LAS(base)* . *LAS(w)* is defined as *E[min(Y, w)]* where *Y* is the severity random variable. If N is the claim count variable, $b_0 = kE(N)*base$.

The ILF is the ratio of limited average severities, so $LAS(x*base) = k*base*x^a$. For simplicity, units are expressed in terms of the base, so $base = 1$ and $LAS(x) = kx^a$ for $x > 1$. If there is a severity distribution *F(y)* underlying this ILF equation, it would satisfy the not-too-hard-to-derive formula:

$$LAS(x) = \int_0^x [1 - F(y)]dy$$

(proved by taking the derivatives of the integral and of the definition of *LAS)*. Then taking the derivative of this formula and of kx^a gives $F(x) = 1 - kax^{a-1}$. This holds at any value *x* for which $LAS(x) = kx^a$. However this formula is normally used for *x* above the base, so it might not hold for smaller *x*. In fact, Mack/Fackler show that it cannot hold for small enough *x*.

This *F* is the simple Pareto if we also impose $x > (ak)^{1/(1-a)}$. For reference, denote this simple Pareto by *G(x)*, i.e., $G(x) = 1 - kax^{a-1}$ for $x > (ak)^{1/(1-a)}$ and zero otherwise. Mack/ Fackler show that *G(x)* does not duplicate the limited average severity *k* at the base. But they also show how it can be used to define a family of severity distributions that does satisfy that condition. In fact they can start with one of many distributions for small claims and find a point *u* below the base for which the starting distribution below *u* combined with *G(x)* above *u* gives a distribution that satisfies these constraints. In many reinsurance applications there is little interest in probabilities very much below the base, so a simple and fairly arbitrary case will illustrate their point.

Begin by defining u_0 as $u_0 = k^{1/(1-a)}$. For $a < 1$, $u_0 > (ak)^{1/(1-a)}$.which is the starting point of G. Then define $F(x)$ by: $F(x) = 0, x < u_0$, $F(x) = G(x)$ for $x \geq u_0$. This has a point mass of $1 - a$ at $x = u_0$. Then for any $x > u_0$, integrating $1 - F(y)$ from 0 to x is $LAS(x)$. From 0 to u_0 the integrand is 1, so evaluates to u_0. From u_0 to x the anti-derivative is ky^a so evaluates to $kx^a - ku_0^a = kx^a - kk^{a/(1-a)} = kx^a - u_0$, so the whole integral $LAS(x)= kx^a$, as required.

Continuous Approach

The mass point can be spread out to produce a continuous distribution. Mack/Fackler show how to do this starting with a distribution for the small claims, say $F(y)$, with parameters to be determined. They define a split-point u that depends on F and the parameter a of G. They show that u is the solution of the equation:

$$aLAS_F(u) = u[1 - F(u)]$$

Then the desired severity distribution is given by:

$F_u(x) = F(x)$ for $x < u$

$F_u(x) = 1 - [1 - F(u)]u^{1-a}x^{a-1} = G(x)$ for $x \geq u$.

To show this, first for F_u so defined, the limited expected value at x is again the integral of $1 - F(y)$ from 0 to x, and again this breaks down as the integral from 0 to u and from u to x. From 0 to u the integral is just $LAS_F(u)$ which is $[1 - F(u)]u/a$ by the definition of u. From u to x the anti-derivative is $[1 - F(u)]u^{1-a}x^a/a$ so the integral can be seen to be:

$[1 - F(u)]u^{1-a}[x^a - u^a]/a.$

This separates into $[1 - F(u)]u^{1-a}x^a/a - [1 - F(u)]u/a$. The second term exactly cancels the integral from 0 to u, so

$$LAS_u(x) = [1 - F(u)]u^{1-a}x^a/a = LAS_F(u)(x/u)^a.$$

This is of the desired form $LAS(x) = kx^a$, with $k = u^{-a}LAS_F(u)$. For a given k, the parameters of F need to be chosen to produce the desired k.

Also, $ak = au^{-a}LAS_F(u) = u^{1-a}[1 - F(u)]$ so $F_u(x) = 1 - akx^{a-1} = G(x)$ for $x \geq u$.

Thus there are two equations to solve for u and a parameter of F:

$$LAS_F(u) = u[1 - F(u)]/a = ku^a.$$

With this solution, the severity distribution F_u is continuous at u but the density is not. Above u the distribution is a simple Pareto, but u is above the minimum point of that Pareto, and F_u agrees with F until $x = u$. Examples are easier to create in those cases where F has closed form limited expected value and distribution functions.

Examples

Take the case where $a = 0.6$, and $k = 0.2$, or 20% of the base. Several forms of the distribution F are illustrated. Note that $G(x) = 1 - akx^{a-1} = 1 - 0.12x^{-.4}$ for $x \geq (ak)^{1/(1-a)} = .00499$ and $u_0 = k^{1/(1-a)} = .01789$. Thus the original discontinuous distribution is $F(x) = 0$ for $x < .01789$ and $F(x) = G(x)$ otherwise.

Exponential

$F(x) = 1 - e^{-x/\theta}$, $LAS(x) = \theta(1 - e^{-x/\theta})$

Thus the system to solve is $\theta(1 - e^{-u/\theta}) = ue^{-u/\theta}/\mathrm{a} = ku^a$

This can be solved for u *and* θ in terms of a and k. Solving simultaneously gives θ = 0.05624 and u = 0.05328. Thus:

$F_u(x) = 1 - e^{-x/0.05624}$, $x < 0.05328$,

$F_u(x) = G(x) = 1 - 0.12x^{-.4}$, $x \geq 0.05328$. Then

$LAS_u(x) = [1 - F(u)]u^{1-a}x^a/a = LAS_F(u)(x/u)^a .=0.2x^{.6}$ as specified with k and a.

Ballasted Pareto

$F(x) = 1 - (1+x/\theta)^{-\alpha}$, $LAS(x) = [\theta/(\alpha - 1)][1 - (1+x/\theta)^{1-\alpha}]$.

This simplifies considerably if just the case $\alpha = 2$ is considered. Then the system to solve becomes $LAS(u) = \theta u/[\theta+u] = u/[a(1+u/\theta)^2] = ku^a$. The middle two lead to $u = \theta(1 - a)/a$. Substituting this for u in the outer two eventually yields $\theta = [a^a/k]^{1/(a-1)}/(1 - a)$. This then plugs in to give $u = [a^a/k]^{1/(a-1)}/a$ and the distribution F. With the selection a = 0.6, k = 0.2, the parameters are θ = 0.09623 and u = 0.06415.

Positive Power Curve

$F(x) = (x/\theta)^\beta$ on $[0, \theta]$, $\beta > 0$, $LAS(x) = x - [\theta/(\beta+1)](x/\theta)^{\beta+1}$. From these the usual derivation gives:

$$u = \left[k\left(1+\frac{1-a}{\beta}\right)\right]^{1/(1-a)} \qquad \theta = u\left[\frac{1+\beta-a}{(1-a)(1+\beta)}\right]^{1/\beta}$$

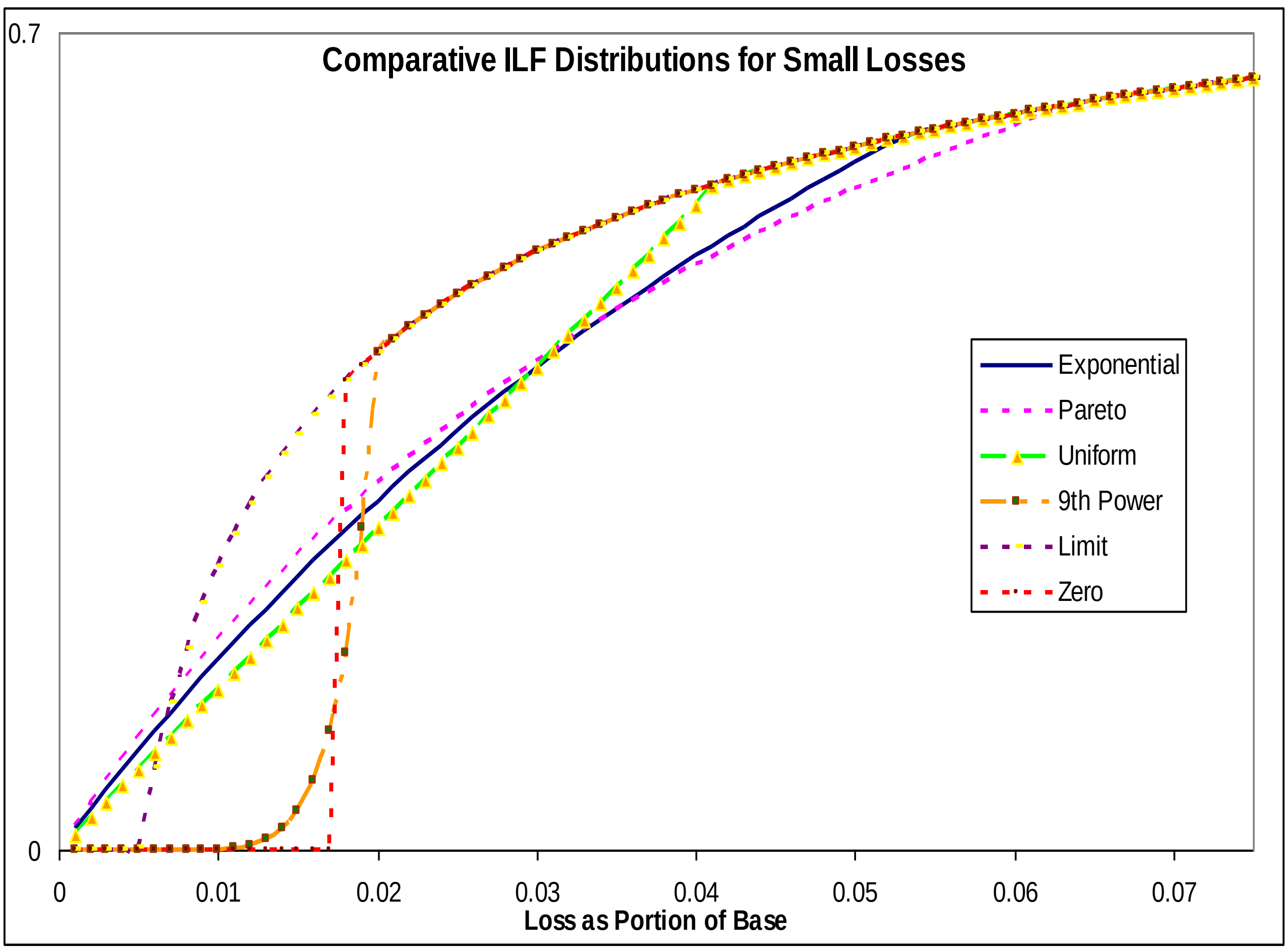
Then $F_u(x) = (x/\theta)^\beta$, $x < u$, and $F_u(x) = G(x)$, $x \geq u$. The simplest special case is β = 1, which makes the interval below the split point uniform on $[0, u]$. In the example a = 0.6, k = 0.2 this gives θ = 0.07260 and u = 0.04149. Thus $F_u(x) = x/.0726$ for $x < 0.04149$ and $F_u(x) = 1 - 0.12x^{-0.4}$ otherwise.

Note that the minimum that u can approach is $u_0 = k^{1/(1-a)}$ as β goes to infinity. Also note that it is possible to start with u and solve backward for the β that gives this u. This is:

$$\beta = \frac{(1-a)k}{u^{1-a} - k}$$

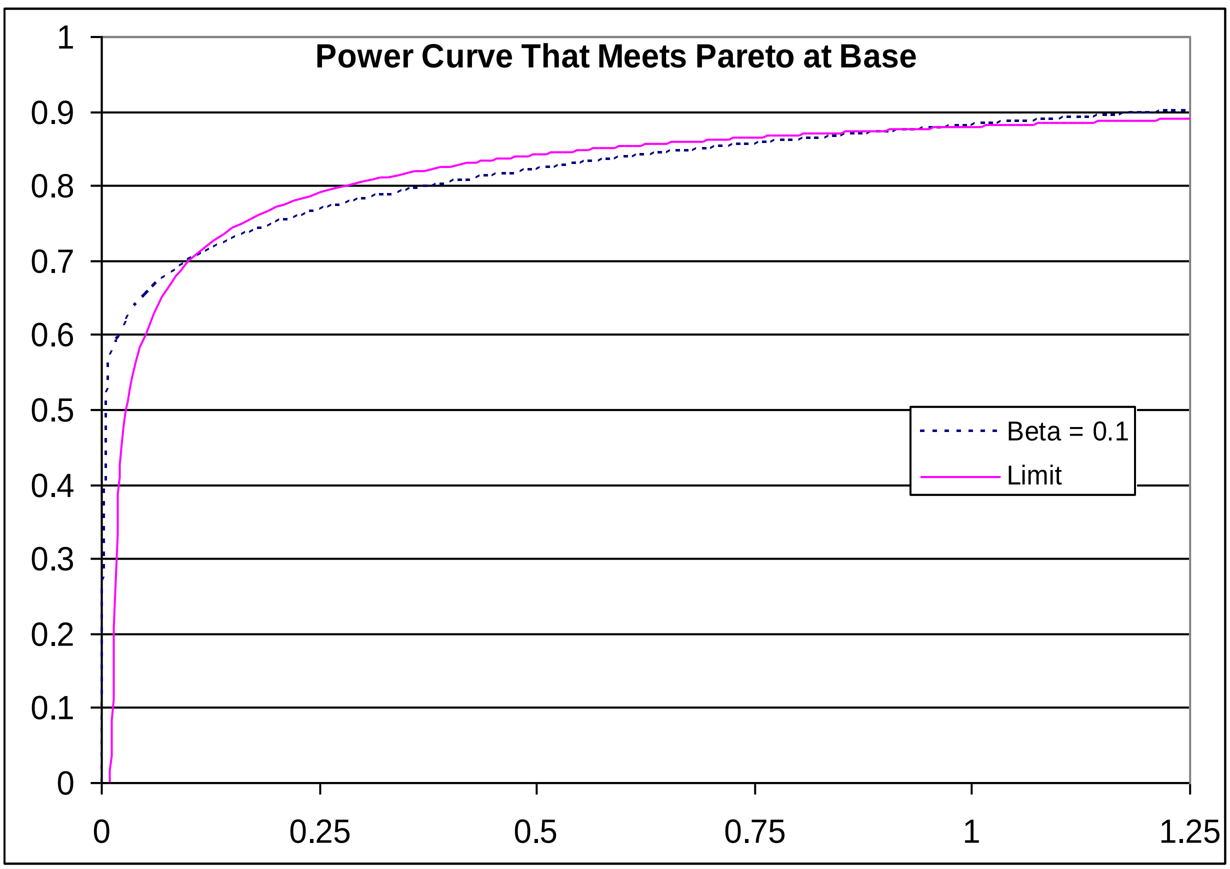
For instance, in the case u=1, so F and G meet at the base, $\beta = (1-a)k/(1-k)$.

The examples are plotted below, along with the power curve with $\beta = 9$ and the limiting simple Pareto.

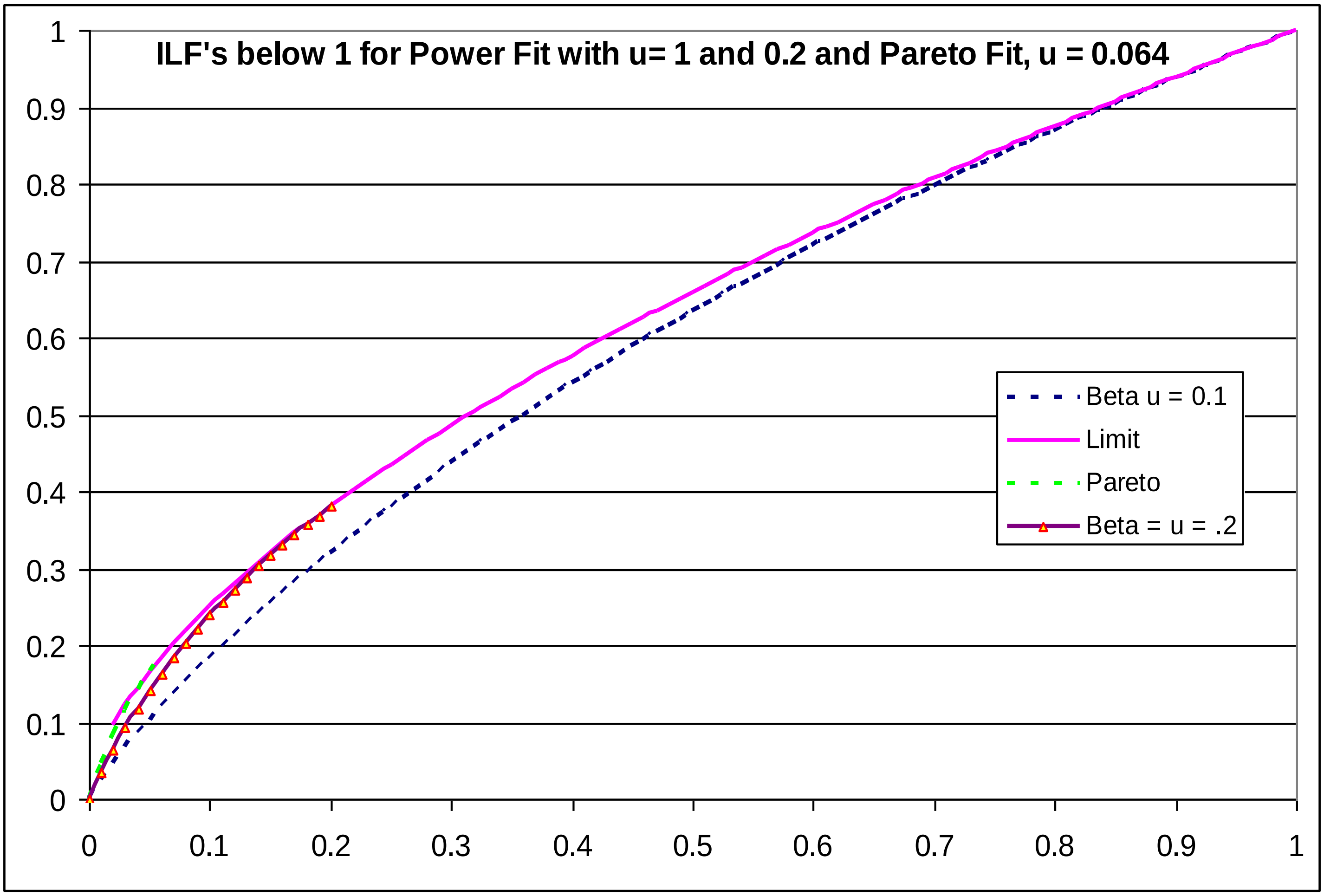


The ballasted Pareto has the least change in slope at the split point and the highest split point. The higher power β gives a continuous approximation to the point mass and has a split point relatively close to u_0. That might be important if the ILF table is used for losses below the base. Limits down to around 2% of the base can use the ILF formula x^a with this power curve or with the non-continuous split.

For a power curve to have $u = 1$, in this case $\beta = 0.1$. This could be of interest if for instance a single curve was desired for all deductibles below the base. This curve and the Pareto limit are graphed up to 1.25 below.



If ILF's are needed below the base, the x^a formula works down to u. Below u the F distribution could be used. The extent to which factors are needed for small losses could influence the choice of F. The increased limits factors for several fits are graphed below. Of these, the Pareto, which has the lowest u, gives the closest



approximation to x^a while the power function with $u = 1$ uses a single curve form below $x = 1$.

Is Riebesell Reasonable?

The Pareto with power parameter $\alpha < 1$ is heavier-tailed than many claims datasets would indicate. Some experienced actuaries tend to believe $\alpha = 1$ is in the right range[1] for liability lines, although some lines could be more heavy-tailed. In addition, ILF's are prices, including loadings, not pure expected losses. Using a more heavy-tailed distribution for pricing is one standard way to

[1] Higher values of α are usually obtained when losses limited by policy limits are included in the fits without adjustment for the effects of policy limits.

incorporate risk loading. Since the Pareto α is $1 - a$, taking a low value of a, like 0.2, would produce an α just somewhat below 1, which would yield a more mild loading for an actual loss distribution with an α of 1, compared to taking a around ½ or greater.

Furthermore, from a Bayesian perspective, if there is a high degree of uncertainty about what the loss distribution is, then the insurer, in terms of current knowledge of the risk, is facing a heavier tailed distribution than ultimate loss results may eventually show.

Another issue is that insureds with greater loss potential may purchase higher limits, which would be consistent with using ILF's that increase more quickly (higher a). In some extreme circumstances increased limits factors give prices per million of coverage that actually increase for higher layers. Thus $a > 1$ in x^a could be practical in some markets, even though this is not consistent with any single loss distribution for all insureds.

Conclusions

The x^a form for ILF's requires a simple Pareto distribution with $\alpha = 1 - a$ above some point $x = u$, with a great deal of flexibility for the distribution F below u. However the ILF's below u do not follow the same formula. For many types of distributions F, it is possible to find u and the parameters of F that accomplish this by solving a system of two equations. If a positive power function is used for the F below u, it is easy to pick u in advance, in the range $1 \geq u > u_0$, and then find the parameters of F.

References

Riebesell, P.(1936) Introduction to Property-Liability Insurance Mathematics (Einführung in die Sachversicherungsmathematik) Berlin.

Mack, T. and Fackler, M. (2003) "Exposure Rating in Liability Reinsurance," Blätter of the DGVFM (German Association for Mathematics of Insurance and Finance), Vol. XXVI, 2.