MMj Copulas

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In search for a wider selection of multivariate copulas, a set of copulas defined by Joe are investigated. While Joe gives a methodology for producing multivariate copulas, only the ones for which he provides closed-form copulas functions are reviewed here. It turns out that these are somewhat restricted in the range of correlations and tail dependencies that can be produced for bivariate pairs within a single multivariate copula. However both right and left positive tail dependence is possible, and the shapes are somewhat different from the t and normal copulas, so potential applicability exists.

MMj Copulas

Joe (1997) provides a methodology for building up multivariate copulas by integrals of simpler copulas. Three of these – called MM1, MM2, and MM3 – have closed form expressions. We investigate some of the properties of these copulas.

NEED FOR MORE MULTIVARIATE COPULAS

A well known multivariate copula is the t-copula. This has a single parameter (called degrees of freedom) that controls the overall tail strength, and for each pair of variables a parameter for the pairwise correlation. This results in giving the more strongly correlated pairs more tail dependency. This is not always appropriate however so the search for more copulas goes on. Joe's MM1, MM2, and MM3 copulas each have an overall strength parameter θ, a parameter δ_{ij} for each pair of variables, and add an additional parameter p_j for each of the m variables U_j, with $1/p_j \geq m - 1$, which gives the possibility of more control over the tails.

The parameters are then δ_{ij} for $i < j$, p_j, for $j = 1, \ldots, m$, and θ. For each variable u_j, it is convenient to use the abbreviations $y_j = (-\ln u_j)^\theta$ and $w_j = p_j(u_j^{-\theta} - 1)$.

MM1

Here $\delta_{ij} \geq 1$ and $\theta \geq 1$. The copula at the m-vector **u** is:

$$C(\mathrm{u}) = \exp\left\{-\left[\sum_{j=1}^{m}(1-(m-1)p_j)y_j + \sum_{i<j}\left(\left(p_i y_i\right)^{\delta_{ij}} + \left(p_j y_j\right)^{\delta_{ij}}\right)^{1/\delta_{ij}}\right]^{1/\theta}\right\}$$

The bivariate i, j margin is:

$$C(u_i,u_j) = \exp\left\{-\left[(1-p_i)y_i + (1-p_j)y_j + \left(\left(p_i y_i\right)^{\delta_{ij}} + \left(p_j y_j\right)^{\delta_{ij}}\right)^{1/\delta_{ij}}\right]^{1/\theta}\right\}$$

Lower tail dependence is zero. Upper tail dependence is given by:

$$\lambda_{ij} = 2 - \left[2 + \left(p_i^{\delta_{ij}} + p_j^{\delta_{ij}}\right)^{1/\delta_{ij}} - p_i - p_j\right]^{1/\theta}$$

MM2

Here $\delta_{ij} > 0$ and $\theta > 0$. The copula at the m-vector **u** is:

$$C(\mathrm{u}) = \left[\sum_{j=1}^{m} u_j^{-\theta} + 1 - m - \sum_{i<j}\left(w_i^{-\delta_{ij}} + w_j^{-\delta_{ij}}\right)^{-1/\delta_{ij}}\right]^{-1/\theta}$$

The i, j margin is:

$$C(u_i,u_j) = \left[u_i^{-\theta} + u_j^{-\theta} - 1 - \left(w_i^{-\delta_{ij}} + w_j^{-\delta_{ij}}\right)^{-1/\delta_{ij}}\right]^{-1/\theta}$$

The upper tail dependence is:

$$\lambda_{ij,U} = \left(p_i^{-\delta_{ij}} + p_j^{-\delta_{ij}}\right)^{-1/\delta_{ij}}$$

The lower tail dependence is:

$$\lambda_{ij,L} = \left[2 - \left(p_i^{-\delta_{ij}} + p_j^{-\delta_{ij}}\right)^{-1/\delta_{ij}}\right]^{-1/\theta}$$

MM3

This starts with $\delta_{ij} > 0$ and $\theta > 1$ (although Joe says 0). The copula at **u** is:

$$C(\mathrm{u}) = \exp\left\{-\left[\sum_{j=1}^{m} y_j - \sum_{i<j}\left(\left(p_i y_i\right)^{-\delta_{ij}} + \left(p_j y_j\right)^{-\delta_{ij}}\right)^{-1/\delta_{ij}}\right]^{1/\theta}\right\}$$

The i,j margin is:

$$C(u_i,u_j) = \exp\left\{-\left[y_i + y_j + \left(\left(p_i y_i\right)^{-\delta_{ij}} + \left(p_j y_j\right)^{-\delta_{ij}}\right)^{-1/\delta_{ij}}\right]^{1/\theta}\right\}$$

The upper tail dependence is:

$$\lambda_{ij} = 2 - \left[2 - \left(p_i^{-\delta_{ij}} + p_j^{-\delta_{ij}}\right)^{-1/\delta_{ij}}\right]^{1/\theta}$$

DENSITIES

It is difficult to write down general formulas for the densities, but some broad outlines can be developed. Recall the product formula for derivatives (ab)′ = a′b+b′a. Then (abc)′ = (ab)′c+abc′ = a′bc+ab′c+abc′. Similarly (abcd)′ = a′bcd+ab′cd+abc′d+abcd′, etc. First consider the simple multivariate function B(**u**) = F(**u**)^a. Denoting partial derivatives by subscripts, we have successively:

$B = F^a$.

$B_i = aF^{a-1}F_i$.

$B_{ij} = a(a-1)F^{a-2}F_iF_j + aF^{a-1}F_{ij}$.

$B_{ijk} = a(a-1)(a-2)F^{a-3}F_iF_jF_k + a(a-1)F^{a-2}(F_iF_{jk} + F_jF_{ik} + F_kF_{ij}) + aF^{a-1}F_{ijk}$.

$B_{ijkl} = a(a-1)(a-2)(a-3)F^{a-4}F_iF_jF_kF_l + a(a-1)(a-2)F^{a-3}(F_{ij}F_kF_l + F_{ik}F_jF_l + F_{il}F_jF_k + F_{jk}F_iF_l + F_{jl}F_iF_k + F_{kl}F_iF_j) + a(a-1)F^{a-2}(F_iF_{jkl} + F_jF_{ikl} + F_kF_{ijl} + F_lF_{ijk} + F_{ij}F_{kl} + F_{ik}F_{jl} + F_{il}F_{jk}) + aF^{a-1}F_{ijkl}$.

This is the form of MM2. Although a pattern is emerging in the subscripts, it seems difficult to write down a general rule for an arbitrary mixed partial.

A similar exercise can be carried out for C(**u**) = $e^{-B(\mathbf{u})}$.

$C = e^{-B}$.

$C_1 = -CB_1$.
$C_{12} = C(B_1B_2 - B_{12})$.
$C_{123} = C(B_1B_{23} + B_2B_{13} + B_3B_{12} - B_1B_2B_3 - B_{123})$.
$C_{1234} = C(B_1B_{234} + B_2B_{134} + B_3B_{124} + B_4B_{123} + B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23} - B_{12}B_3B_4 - B_{13}B_2B_4 - B_{14}B_2B_3 - B_{23}B_1B_4 - B_{24}B_1B_3 - B_{34}B_1B_2 + B_1B_2B_3B_4 - B_{1234})$.

This is the form of MM1 and MM3 with F^a substituted in for B.
To calculate the derivatives let $x_j = y_j' = \theta(-\ln u_j)^{\theta-1}/u_j$, and $t_j = w_j' = \theta p_j u_j^{-\theta-1}$. The case m=3 is not too bad. First for MM1:

$$F(\mathrm{u}) = \sum_{j=1}^{3}(1-2p_j)y_j + \sum_{i<j}\left(\left(p_i y_i\right)^{\delta_{ij}} + \left(p_j y_j\right)^{\delta_{ij}}\right)^{1/\delta_{ij}}$$

Taking the first derivative:

$$F_i(\mathrm{u}) = (1-2p_j)y_j x_i + (p_i y_i)^{\delta_{ij}-1} p_i x_i \sum_{j\neq i}\left(\left(p_i y_i\right)^{\delta_{ij}} + \left(p_j y_j\right)^{\delta_{ij}}\right)^{1/\delta_{ij}-1}$$

This would be almost the same for the bivariate margin. Then:

$$F_{ij}(\mathrm{u}) = (1-\delta_{ij})(p_i p_j y_i y_j)^{\delta_{ij}-1} p_i p_j x_i x_j \left(\left(p_i y_i\right)^{\delta_{ij}} + \left(p_j y_j\right)^{\delta_{ij}}\right)^{1/\delta_{ij}-2}$$

This is the same for the bivariate margin. From this it is clear that $F_{123} = 0$. Then taking $a = 1/\theta$ and plugging in these values of F_i and F_{ij} will give all values of B_i, B_{ij}, and B_{123}, which can then be plugged in the formula for C_{123} to give the density. From the formulas for C_{12} the bivariate density is similar.

MM2 is even easier, as the C formulas are not needed. What is needed is:

$$F(\mathrm{u}) = \sum_{j=1}^{3} u_j^{-\theta} - 2 - \sum_{i<j}\left(w_i^{-\delta_{ij}} + w_j^{-\delta_{ij}}\right)^{-1/\delta_{ij}}$$

This gives:

$$F_i(\mathrm{u}) = -\theta u_i^{-\theta-1} - w_i^{-\delta_{ij}-1} t_i \sum_{j\neq i}\left(w_i^{-\delta_{ij}} + w_j^{-\delta_{ij}}\right)^{-1/\delta_{ij}-1}$$

And:

$$F_{ij}(\mathrm{u}) = -(1+\delta_{ij})(w_i w_j)^{-\delta_{ij}-1} t_i t_j \left(w_i^{-\delta_{ij}} + w_j^{-\delta_{ij}}\right)^{-1/\delta_{ij}-2}$$

Again $F_{123} = 0$. Then setting $a = -1/\theta$ gives B_{123} which is the density.

RANGE OF POSSIBLE CORRELATIONS AND DEPENDENCIES

It turns out that not every possible correlation matrix can be input into these copulas. The θ parameter determines a lot of what is possible for each copula. The tables show the tail dependence and Spearman's ρ for selected parameters.

MM1 Upper Tail Dependence

δ	θ:	1.037	1.037	1.037	1.037	1.11	1.11	1.11	1.11	1.33	1.33	1.33	1.33	2	2	2	2	4	4	4	4
	p_i p_j:	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5
1.01	0.005	5%	5%	5%	5%	13%	13%	13%	13%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
1.01	0.17	5%	5%	5%	5%	13%	14%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
1.01	0.335	5%	5%	5%	5%	13%	14%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
1.01	0.5	5%	5%	5%	6%	13%	14%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
1.1	0.005	5%	5%	5%	5%	13%	14%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
1.1	0.17	5%	7%	8%	8%	14%	15%	16%	16%	32%	33%	34%	34%	59%	59%	60%	60%	81%	81%	82%	82%
1.1	0.335	5%	8%	9%	10%	14%	16%	17%	18%	32%	34%	34%	35%	59%	60%	60%	60%	81%	82%	82%	82%
1.1	0.5	5%	8%	10%	11%	14%	16%	18%	19%	32%	34%	35%	36%	59%	60%	60%	61%	81%	82%	82%	82%
2	0.005	5%	5%	5%	5%	14%	14%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
2	0.17	5%	14%	17%	18%	14%	22%	24%	25%	32%	38%	40%	41%	59%	62%	63%	64%	81%	83%	83%	83%
2	0.335	5%	17%	23%	27%	14%	24%	30%	33%	32%	40%	44%	47%	59%	63%	66%	67%	81%	83%	84%	85%
2	0.5	5%	18%	27%	33%	14%	25%	33%	38%	32%	41%	47%	51%	59%	64%	67%	69%	81%	83%	85%	86%
11	0.005	5%	5%	5%	5%	14%	14%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
11	0.17	5%	20%	21%	21%	14%	27%	28%	28%	32%	42%	43%	43%	59%	64%	65%	65%	81%	84%	84%	84%
11	0.335	5%	21%	34%	36%	14%	28%	40%	42%	32%	43%	52%	53%	59%	65%	70%	71%	81%	84%	86%	86%
11	0.5	5%	21%	36%	49%	14%	28%	42%	53%	32%	43%	53%	62%	59%	65%	71%	76%	81%	84%	86%	89%
101	0.005	5%	5%	5%	5%	14%	14%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
101	0.17	5%	21%	21%	21%	14%	28%	28%	28%	32%	43%	43%	43%	59%	65%	65%	65%	81%	84%	84%	84%
101	0.335	5%	21%	36%	37%	14%	28%	42%	42%	32%	43%	53%	53%	59%	65%	71%	71%	81%	84%	86%	86%
101	0.5	5%	21%	37%	52%	14%	28%	42%	56%	32%	43%	53%	64%	59%	65%	71%	77%	81%	84%	86%	89%

MM1 Spearman's ρ

δ	θ:	1.037	1.037	1.037	1.037	1.11	1.11	1.11	1.11	1.33	1.33	1.33	1.33	2	2	2	2	4	4	4	4
	p_i p_j:	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5
1.01	0.005	4%	4%	4%	4%	10%	10%	10%	10%	25%	25%	25%	25%	50%	50%	50%	50%	75%	75%	75%	75%
1.01	0.17	4%	4%	4%	4%	10%	10%	10%	10%	25%	25%	25%	25%	50%	50%	50%	50%	75%	75%	75%	75%
1.01	0.335	4%	4%	4%	4%	10%	10%	10%	10%	25%	25%	25%	25%	50%	50%	50%	50%	75%	75%	75%	75%
1.01	0.5	4%	4%	4%	4%	10%	10%	10%	10%	25%	25%	25%	25%	50%	50%	50%	50%	75%	75%	75%	75%
1.1	0.005	4%	4%	4%	4%	10%	10%	10%	10%	25%	25%	25%	25%	50%	50%	50%	50%	75%	75%	75%	75%
1.1	0.17	4%	5%	6%	6%	10%	11%	12%	12%	25%	26%	27%	27%	50%	51%	51%	51%	75%	76%	76%	76%
1.1	0.335	4%	6%	6%	7%	10%	12%	13%	13%	25%	27%	27%	28%	50%	51%	52%	52%	75%	76%	76%	76%
1.1	0.5	4%	6%	7%	8%	10%	12%	13%	14%	25%	27%	28%	28%	50%	51%	52%	52%	75%	76%	76%	76%
2	0.005	4%	4%	4%	4%	10%	10%	10%	10%	25%	25%	25%	25%	50%	50%	50%	50%	75%	75%	75%	75%
2	0.17	4%	10%	12%	14%	10%	16%	18%	20%	25%	30%	32%	33%	50%	53%	55%	55%	75%	77%	78%	78%
2	0.335	4%	12%	17%	20%	10%	18%	22%	25%	25%	32%	35%	38%	50%	55%	57%	59%	75%	78%	79%	80%
2	0.5	4%	14%	20%	24%	10%	20%	25%	29%	25%	33%	38%	41%	50%	55%	59%	61%	75%	78%	80%	81%
11	0.005	4%	4%	4%	4%	10%	10%	10%	10%	25%	25%	25%	25%	50%	50%	50%	50%				
11	0.17	4%	13%	16%	17%	10%	19%	21%	23%	25%	33%	34%	36%	50%	56%	56%	57%				
11	0.335	4%	16%	24%	27%	10%	21%	29%	32%	25%	34%	41%	43%	50%	56%	63%	61%				
11	0.5	4%	17%	27%	36%	10%	23%	32%	41%	25%	36%	43%	52%	50%	57%	61%	71%				

MM2 Tail Dependence Upper Lower

δ	p_i	Upper θ: any	any	any	any	Lower 0.11	0.11	0.11	0.11	0.33	0.33	0.33	0.33	1	1	1	1	3	3	3	3
	p_j:	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5
0.25	0.005	0%	0%	0%	0%	0%	0%	0%	0%	13%	13%	13%	13%	50%	50%	50%	50%	79%	79%	79%	79%
0.25	0.17	0%	1%	1%	2%	0%	0%	0%	0%	13%	13%	13%	13%	50%	50%	50%	50%	79%	80%	80%	80%
0.25	0.335	0%	1%	2%	3%	0%	0%	0%	0%	13%	13%	13%	13%	50%	50%	51%	51%	79%	80%	80%	80%
0.25	0.5	0%	2%	3%	3%	0%	0%	0%	0%	13%	13%	13%	13%	50%	50%	51%	51%	79%	80%	80%	80%
1	0.005	0%	0%	0%	0%	0%	0%	0%	0%	13%	13%	13%	13%	50%	50%	50%	50%	79%	79%	79%	79%
1	0.17	0%	9%	11%	13%	0%	0%	0%	0%	13%	14%	15%	15%	50%	52%	53%	53%	79%	81%	81%	81%
1	0.335	0%	11%	17%	20%	0%	0%	0%	1%	13%	15%	16%	17%	50%	53%	55%	56%	79%	81%	82%	82%
1	0.5	0%	13%	20%	25%	0%	0%	1%	1%	13%	15%	17%	19%	50%	53%	56%	57%	79%	81%	82%	83%
4	0.005	0%	0%	0%	0%	0%	0%	0%	0%	13%	13%	13%	13%	50%	50%	50%	50%	79%	79%	79%	79%
4	0.17	0%	14%	17%	17%	0%	0%	0%	0%	13%	16%	16%	16%	50%	54%	55%	55%	79%	81%	82%	82%
4	0.335	0%	17%	28%	32%	0%	0%	1%	1%	13%	16%	20%	21%	50%	55%	58%	60%	79%	82%	83%	84%
4	0.5	0%	17%	32%	42%	0%	0%	1%	2%	13%	16%	21%	25%	50%	55%	60%	63%	79%	82%	84%	86%
16	0.005	0%	1%	1%	1%	0%	0%	0%	0%	13%	13%	13%	13%	50%	50%	50%	50%	79%	79%	79%	79%
16	0.17	1%	16%	17%	17%	0%	0%	0%	0%	13%	16%	16%	16%	50%	54%	55%	55%	79%	82%	82%	82%
16	0.335	1%	17%	32%	33%	0%	0%	1%	1%	13%	16%	21%	22%	50%	55%	60%	60%	79%	82%	84%	84%
16	0.5	1%	17%	33%	48%	0%	0%	1%	2%	13%	16%	22%	28%	50%	55%	60%	66%	79%	82%	84%	87%
64	0.005	0%	1%	1%	1%	0%	0%	0%	0%	13%	13%	13%	13%	50%	50%	50%	50%	79%	79%	79%	79%
64	0.17	1%	17%	17%	17%	0%	0%	0%	0%	13%	16%	16%	16%	50%	55%	55%	55%	79%	82%	82%	82%
64	0.335	1%	17%	33%	33%	0%	0%	1%	1%	13%	16%	22%	22%	50%	55%	60%	60%	79%	82%	84%	84%
64	0.5	1%	17%	33%	49%	0%	0%	1%	3%	13%	16%	22%	29%	50%	55%	60%	66%	79%	82%	84%	87%

MM2 Spearman's ρ

δ	p_i	θ: 0.037	0.037	0.037	0.037	0.11	0.11	0.11	0.11	0.33	0.33	0.33	0.33	1	1	1	1	3	3	3	3
	p_j:	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5
0.25	0.005	3%	3%	3%	3%	8%	8%	8%	8%	21%	21%	21%	21%	48%	48%	48%	48%	79%	79%	79%	79%
0.25	0.17	3%	4%	4%	5%	8%	9%	9%	10%	21%	22%	23%	23%	48%	49%	49%	49%	79%	79%	79%	79%
0.25	0.335	3%	4%	5%	6%	8%	9%	10%	11%	21%	23%	23%	24%	48%	49%	49%	50%	79%	79%	79%	79%
0.25	0.5	3%	5%	6%	6%	8%	10%	11%	11%	21%	23%	24%	24%	48%	49%	50%	50%	79%	79%	79%	80%
1	0.005	3%	3%	3%	3%	8%	8%	9%	9%	21%	22%	22%	22%	48%	48%	48%	48%	79%	79%	79%	79%
1	0.17	3%	11%	14%	16%	8%	16%	19%	21%	22%	28%	31%	33%	48%	53%	55%	56%	79%	81%	82%	82%
1	0.335	3%	14%	20%	24%	9%	19%	24%	28%	22%	31%	36%	39%	48%	55%	58%	60%	79%	82%	83%	84%
1	0.5	3%	16%	24%	29%	9%	21%	28%	33%	22%	33%	39%	43%	48%	56%	60%	63%	79%	82%	84%	85%
4	0.005	3%	3%	3%	3%	8%	9%	9%	9%	22%	22%	22%	22%	48%	48%	48%	48%	79%	79%	79%	79%
4	0.17	3%	15%	20%	22%	9%	20%	24%	26%	22%	32%	35%	37%	48%	55%	57%	59%	79%	82%	83%	83%
4	0.335	3%	20%	28%	34%	9%	24%	32%	38%	22%	35%	42%	47%	48%	57%	62%	65%	79%	83%	85%	86%
4	0.5	3%	22%	34%	42%	9%	26%	38%	46%	22%	37%	47%	54%	48%	59%	65%	70%	79%	83%	86%	88%
16	0.005	3%	3%	3%	3%	8%	9%	9%	9%	22%	22%	22%	22%	48%	48%	48%	48%	79%	79%	79%	79%
16	0.17	3%	16%	20%	23%	9%	20%	25%	27%	22%	32%	36%	38%	48%	55%	58%	59%	79%	82%	83%	83%
16	0.335	3%	20%	29%	35%	9%	25%	33%	39%	22%	36%	43%	48%	48%	58%	63%	66%	79%	83%	85%	86%
16	0.5	3%	23%	35%	44%	9%	27%	39%	47%	22%	38%	48%	55%	48%	59%	66%	71%	79%	83%	86%	88%

MM3 Upper Tail Dependence

δ	θ:	1.037	1.037	1.037	1.037	1.11	1.11	1.11	1.11	1.33	1.33	1.33	1.33	2	2	2	2	4	4	4	4
	p_i p_j:	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5
0.25	0.005	5%	5%	5%	5%	13%	13%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
0.25	0.17	5%	6%	6%	7%	13%	14%	15%	15%	32%	32%	33%	33%	59%	59%	59%	59%	81%	81%	81%	81%
0.25	0.335	5%	6%	7%	7%	14%	15%	15%	16%	32%	33%	33%	33%	59%	59%	59%	59%	81%	81%	81%	81%
0.25	0.5	5%	7%	7%	8%	14%	15%	16%	16%	32%	33%	33%	34%	59%	59%	59%	60%	81%	81%	81%	82%
1	0.005	5%	5%	5%	5%	14%	14%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
1	0.17	5%	13%	16%	17%	14%	21%	23%	24%	32%	37%	39%	40%	59%	62%	63%	63%	81%	82%	83%	83%
1	0.335	5%	16%	21%	24%	14%	23%	28%	30%	32%	39%	42%	45%	59%	63%	65%	66%	81%	83%	84%	84%
1	0.5	5%	17%	24%	28%	14%	24%	30%	35%	32%	40%	45%	48%	59%	63%	66%	68%	81%	83%	84%	85%
4	0.005	5%	5%	5%	5%	14%	14%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
4	0.17	5%	18%	21%	21%	14%	25%	28%	28%	32%	41%	42%	43%	59%	64%	65%	65%	81%	83%	84%	84%
4	0.335	5%	21%	31%	35%	14%	28%	37%	40%	32%	42%	50%	52%	59%	65%	69%	70%	81%	84%	86%	86%
4	0.5	5%	21%	35%	45%	14%	28%	40%	49%	32%	43%	52%	59%	59%	65%	70%	74%	81%	84%	86%	88%
16	0.005	5%	5%	5%	5%	14%	14%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
16	0.17	5%	20%	21%	21%	14%	27%	28%	28%	32%	42%	43%	43%	59%	64%	65%	65%	81%	84%	84%	84%
16	0.335	5%	21%	35%	37%	14%	28%	41%	42%	32%	43%	52%	53%	59%	65%	70%	71%	81%	84%	86%	86%
16	0.5	5%	21%	37%	50%	14%	28%	42%	54%	32%	43%	53%	63%	59%	65%	71%	77%	81%	84%	86%	89%
64	0.005	5%	5%	5%	5%	14%	14%	14%	14%	32%	32%	32%	32%	59%	59%	59%	59%	81%	81%	81%	81%
64	0.17	5%	21%	21%	21%	14%	28%	28%	28%	32%	43%	43%	43%	59%	65%	65%	65%	81%	84%	84%	84%
64	0.335	5%	21%	36%	37%	14%	28%	41%	42%	32%	43%	53%	53%	59%	65%	71%	71%	81%	84%	86%	86%
64	0.5	5%	21%	37%	52%	14%	28%	42%	55%	32%	43%	53%	64%	59%	65%	71%	77%	81%	84%	86%	89%

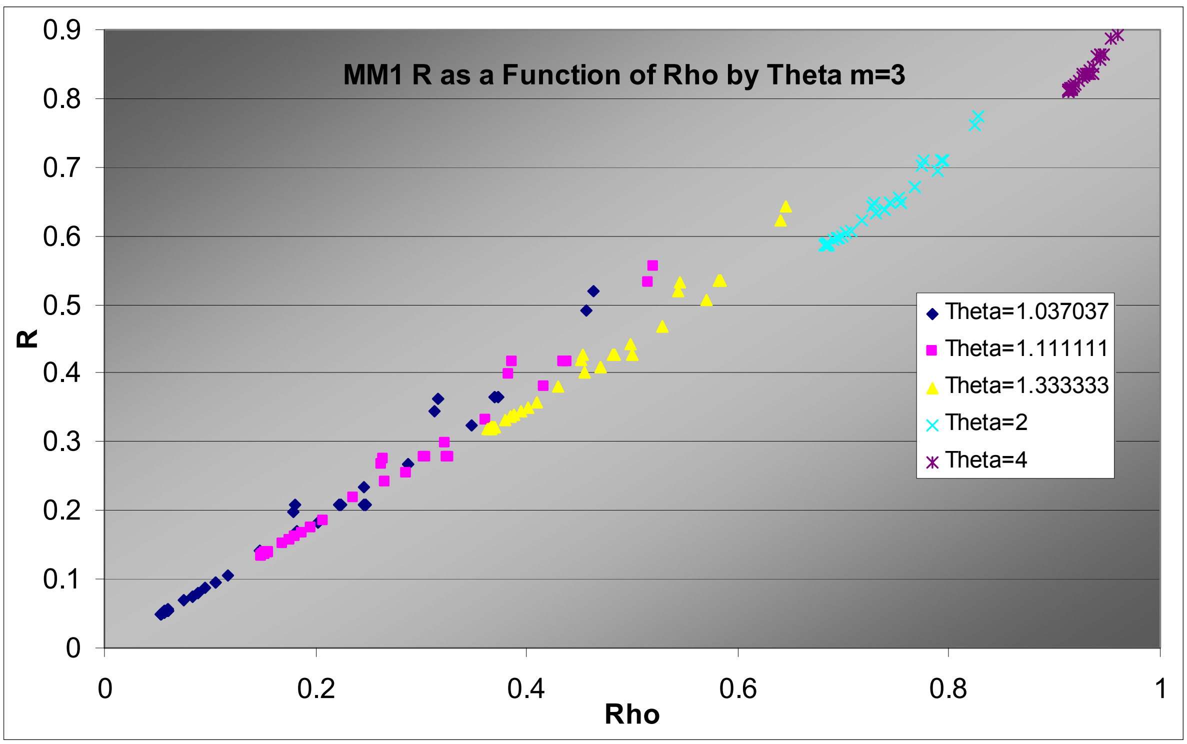
MM3 Spearman's ρ

δ	θ:	1.037	1.037	1.037	1.037	1.11	1.11	1.11	1.11	1.33	1.33	1.33	1.33	2	2	2	2	4	4	4	4
	p_i p_j:	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5	0.005	0.17	0.335	0.5
0.25	0.005	5%	5%	5%	5%	15%	15%	15%	15%	36%	36%	36%	36%	68%	68%	68%	68%	91%	91%	91%	91%
0.25	0.17	5%	4%	4%	3%	15%	14%	13%	13%	36%	36%	35%	35%	68%	68%	68%	67%	91%	91%	91%	91%
0.25	0.335	5%	4%	3%	3%	15%	13%	13%	12%	36%	35%	35%	34%	68%	68%	67%	67%	91%	91%	91%	91%
0.25	0.5	5%	3%	3%	2%	15%	13%	12%	12%	36%	35%	34%	34%	68%	67%	67%	67%	91%	91%	91%	91%
1	0.005	5%	5%	5%	5%	15%	14%	14%	14%	36%	36%	36%	36%	68%	68%	68%	68%	91%	91%	91%	91%
1	0.17	5%	-3%	-5%	-7%	14%	8%	5%	4%	36%	31%	29%	27%	68%	65%	64%	63%	91%	90%	90%	90%
1	0.335	5%	-5%	-10%	-13%	14%	5%	1%	-2%	36%	29%	25%	23%	68%	64%	62%	61%	91%	90%	90%	89%
1	0.5	5%	-7%	-13%	-17%	14%	4%	-2%	-6%	36%	27%	23%	20%	68%	63%	61%	60%	91%	90%	89%	89%
4	0.005	5%	5%	5%	5%	15%	14%	14%	14%	36%	36%	36%	36%	68%	68%	68%	68%	91%	91%	91%	91%
4	0.17	5%	-6%	-9%	-11%	14%	5%	1%	0%	36%	28%	26%	24%	68%	64%	63%	62%	91%	90%	90%	89%
4	0.335	5%	-9%	-16%	-20%	14%	1%	-5%	-8%	36%	26%	21%	18%	68%	63%	60%	59%	91%	90%	89%	89%
4	0.5	5%	-11%	-20%	-25%	14%	0%	-8%	-13%	36%	24%	18%	14%	68%	62%	59%	57%	91%	89%	89%	88%
16	0.005	5%	5%	5%	5%	15%	14%	14%	14%	36%	36%	36%	36%	68%	68%	68%	68%	91%	91%	91%	91%
16	0.17	5%	-6%	-10%	-12%	14%	4%	1%	-1%	36%	28%	25%	24%	68%	64%	63%	62%	91%	90%	90%	89%
16	0.335	5%	-10%	-17%	-21%	14%	1%	-5%	-9%	36%	25%	21%	18%	68%	63%	60%	58%	91%	90%	89%	88%
16	0.5	5%	-12%	-21%	-26%	14%	-1%	-9%	-14%	36%	24%	18%	14%	68%	62%	58%	56%	91%	89%	88%	88%

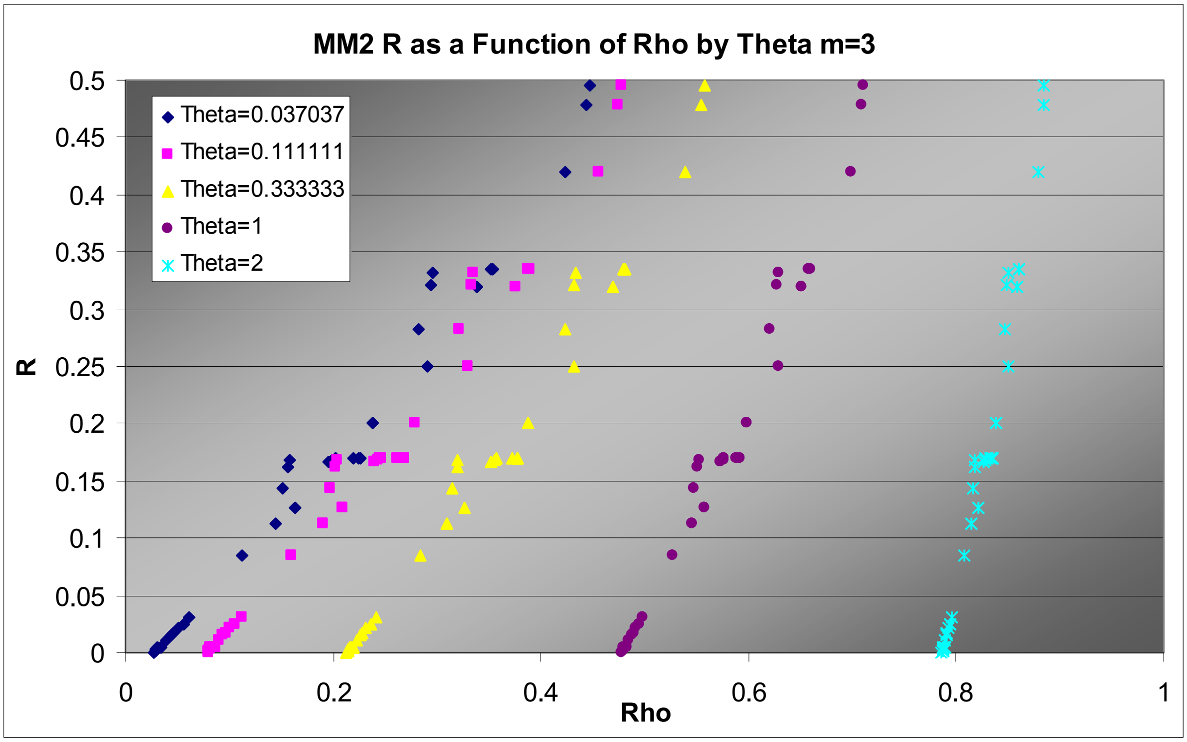
DISCUSSION

The tables above are bivariate relationships. However since the p parameters can be no greater than 1/(m-1), the possible values of p, and so the range of possible correlations and tail dependencies, reduce as the dimension increases. The θ parameter in general has a great deal of influence over what any of these copulas can do. As it gets higher, the other parameters have very little influence at all. Thus when it is high, all pairs of variables will have similar correlations and tail dependencies. When θ is low, however, it is not possible to have very high correlations. Thus in general these copulas can only apply when all pairs of variables have similar tails and correlations.

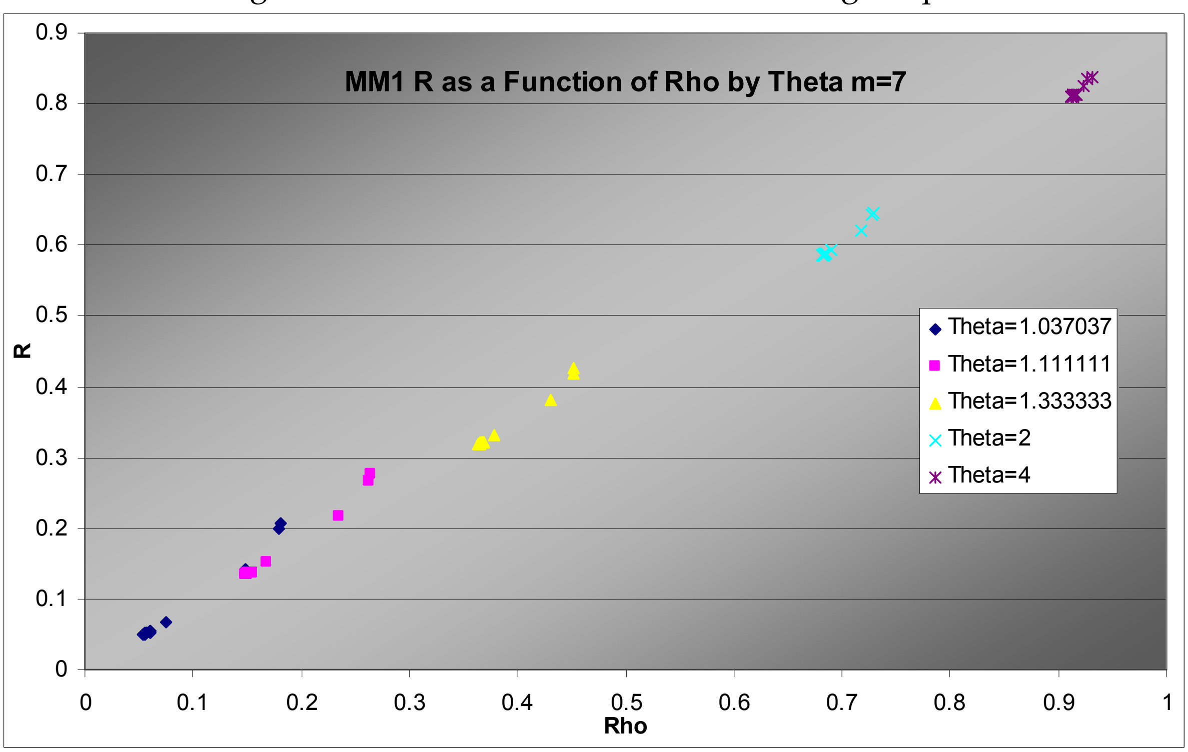
Also the right tail dependency R and the correlation are closely related. Scatter

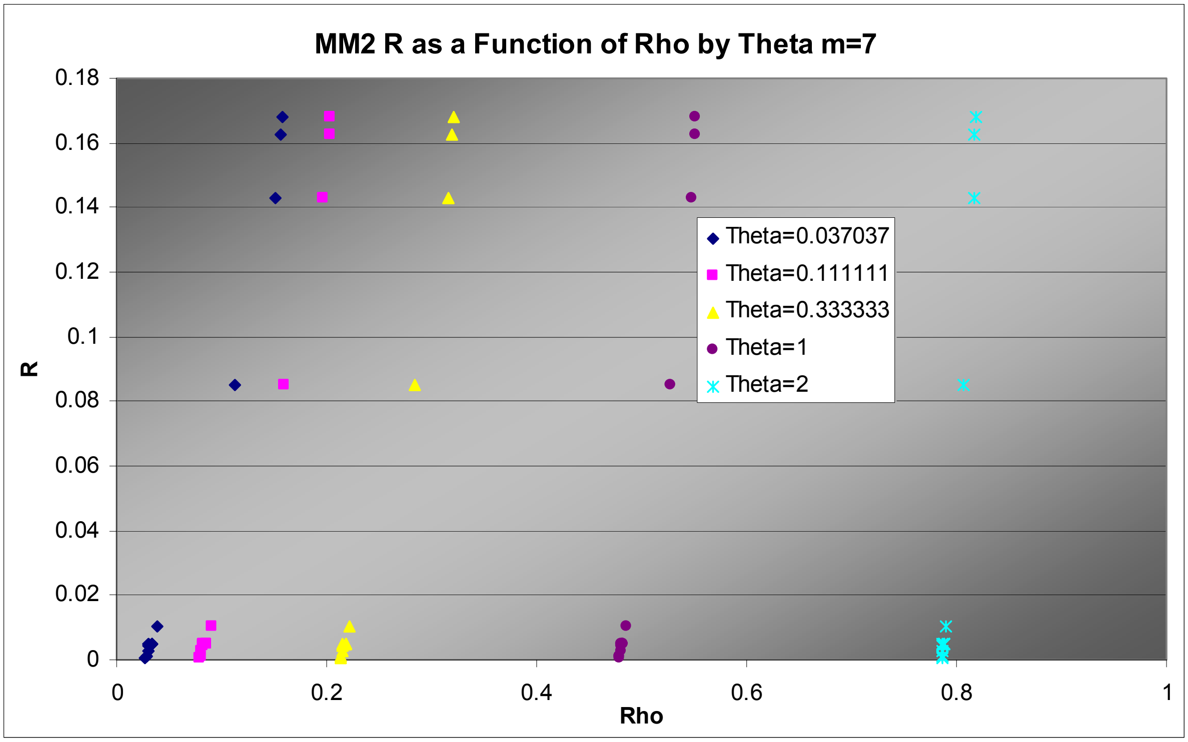


plots of R as a function of ρ for different values of θ are shown for the cases m=3 and m=7 for the MM1 and MM2 copulas.

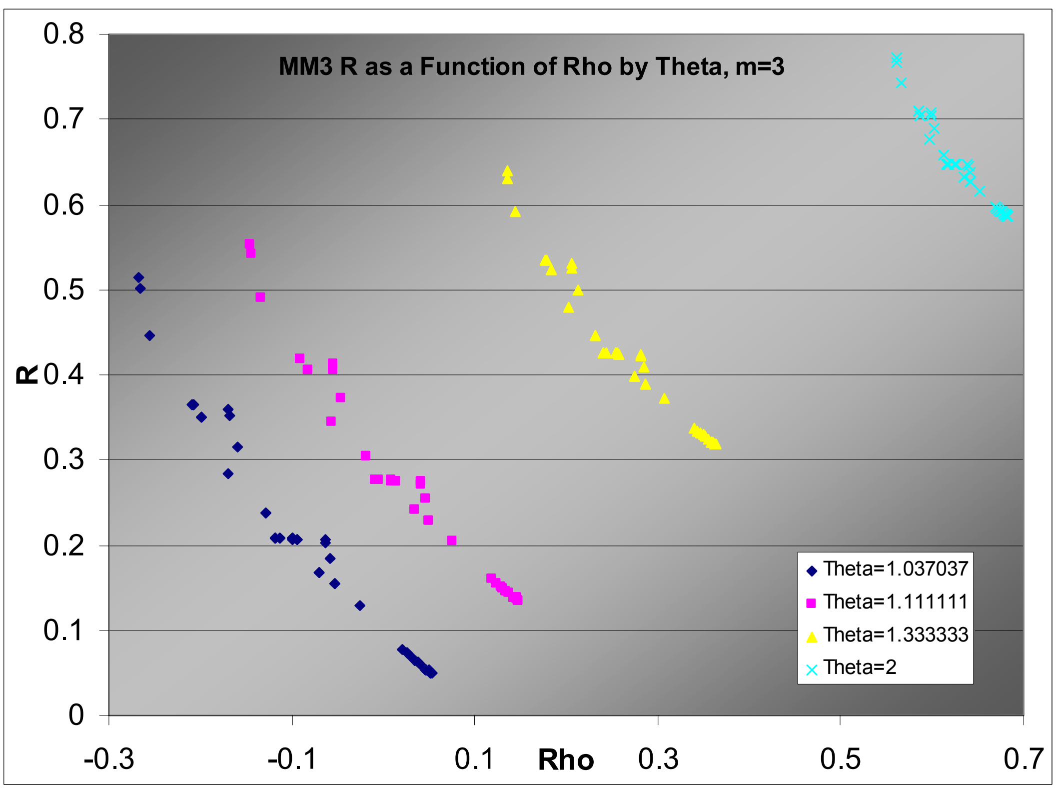


For MM2, R is not a function of θ, so a wider range of R's is possible for any θ. The possible range of ρ for eachθ is about the same for each copula. This range decreases with higher m's however due to the restrict range of p.

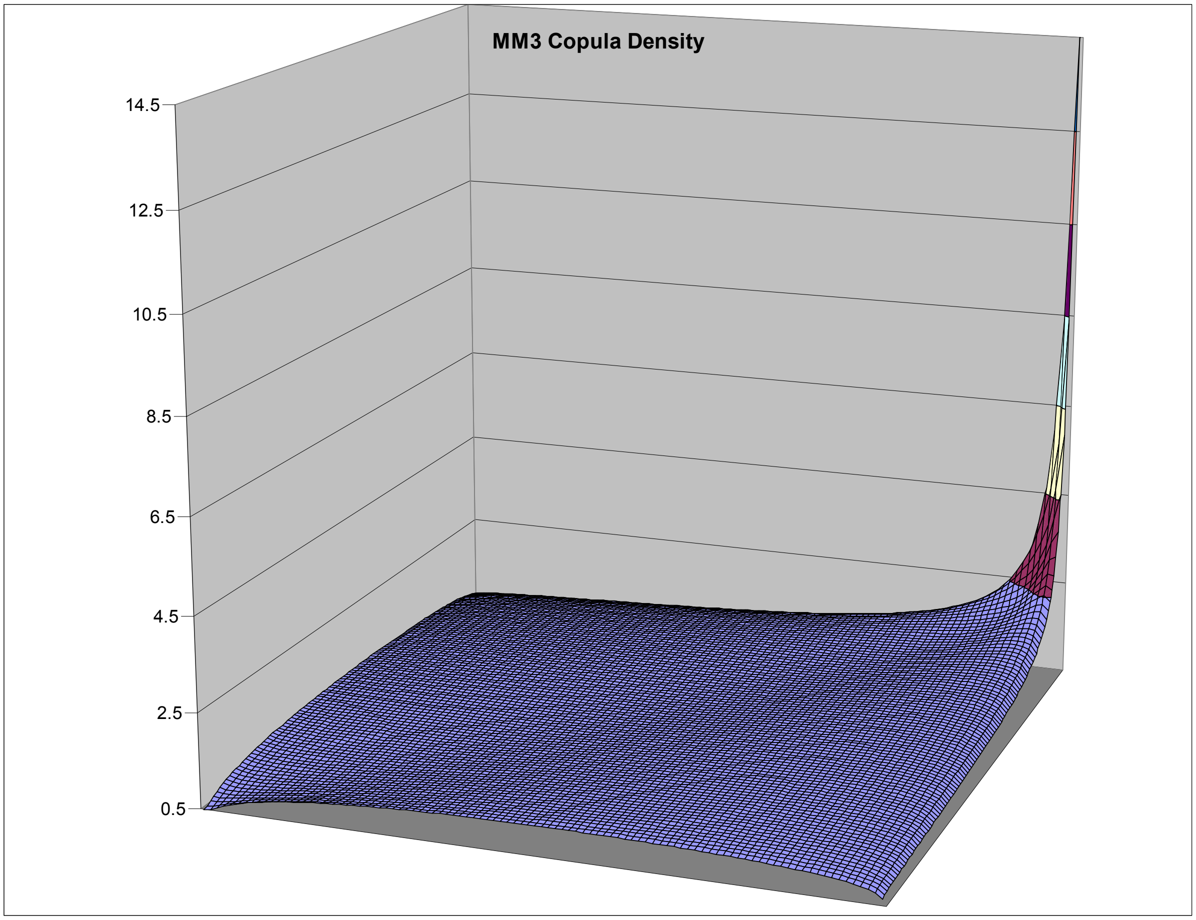




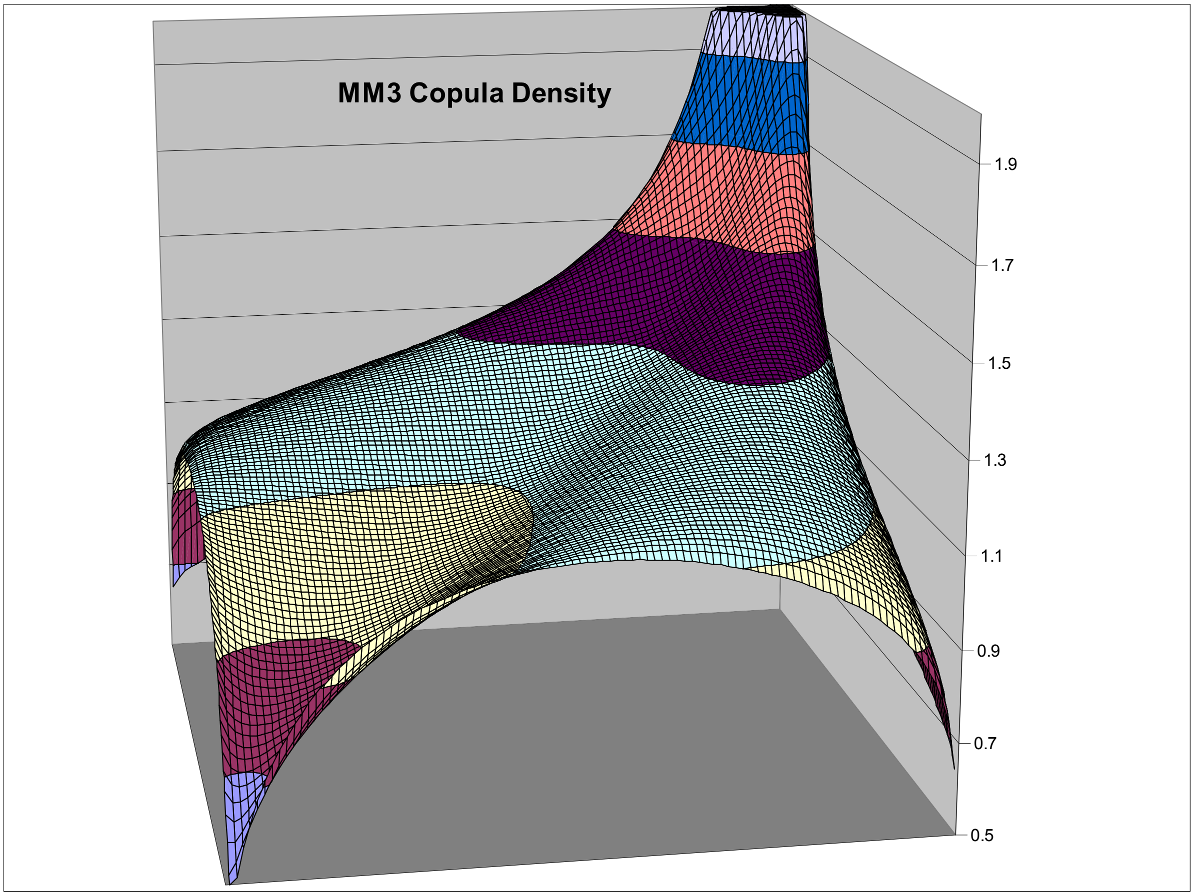
MM3 is somewhat unusual in that R decreases as ρ increases for fixed θ.



The correlation can even be negative with positive R. The copula density helps show how that happens, in this case shown for $<\theta,\delta,p_1,p_2,> = <1.125, 4, \frac{1}{2} , \frac{1}{2} >$.



The overall view shows the tail dependence, which is about R = 50%. Capping the density at 2 shows the finer structure, which features relatively low values around the main diagonal. In this case this leads to an overall rank correlation of about ρ = -11%.



FITTING

We tried fitting the MM1 and MM2 copulas to currency rate changes that previously were fit fairly reasonably by the t-copula. The data consists of monthly changes in the US $ exchange rate for the Swedish, Japanese and Canadian currencies from 1971 through September 2005.

The MMj copula densities get increasingly difficult to calculate as the dimension increases. For this reason some alternatives to MLE were explored. One alternative was the product of the bivariate likelihood functions, which just requires the bivariate density. Since the copulas are defined by the copula functions, an even easier fit is to minimize the distance between the empirical and fitted copulas. Numerical differentiation is discussed in the Appendix, but wasn't tried here.

The correlation of the Swedish and Japanese currencies was fairly high, while their correlations with Canada were both quite a bit lower. This range of correlations is difficult for the MMj copulas, which as seen above require similar correlations for all pairs. The fitted and empirical Spearman ρ correlations are:

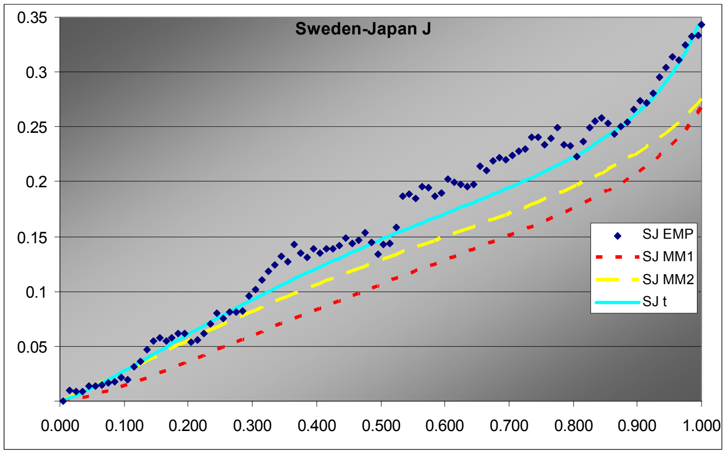
	ρ - SJ	ρ - SC	ρ - JC
Empirical	0.562	0.323	0.166
MM1 mle tri:	0.395	0.219	0.163
MM1 mle bi:	0.411	0.257	0.166
MM1 sse:	0.466	0.266	0.114
MM2 mle bi:	0.393	0.235	0.149
MM 2 sse:	0.471	0.274	0.133
t mle:	0.567	0.344	0.184

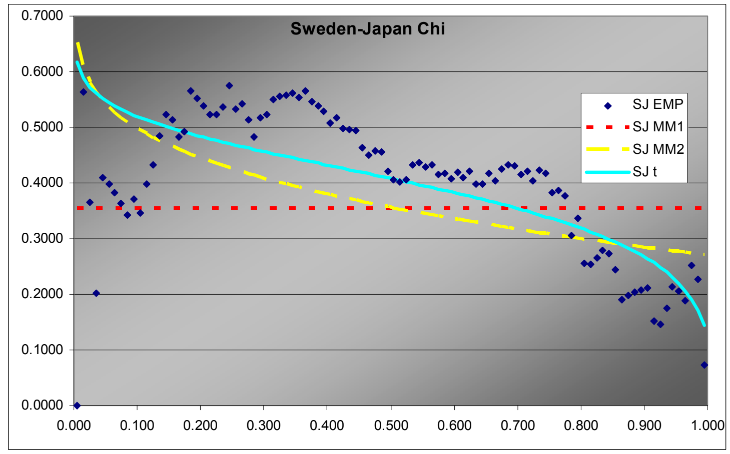
Interestingly, the SSE fits actually came closer to the empirical correlations than did MLE. Other tests of SSE fits were not as good, however. The t-copula can take an arbitrary correlation matrix, so had no trouble with the ρ's.

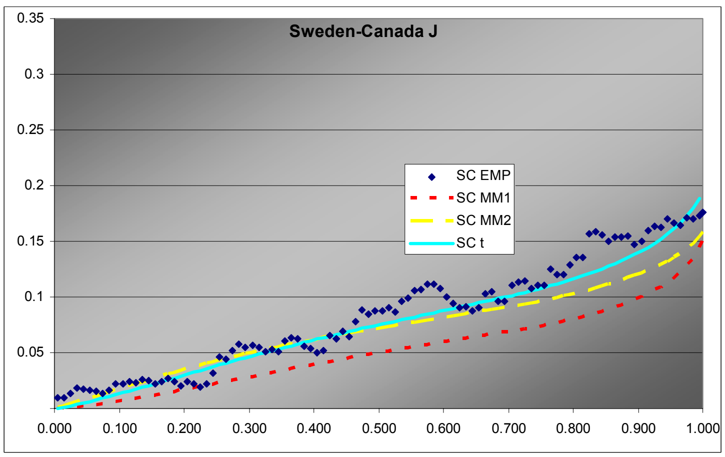
The product of the bivariate MLE estimates for MM1 were quite close to the full trivariate MLE. It should be noted however that there are a lot of local maximums for both likelihood functions, so we cannot be absolutely sure these are the global maximums. Only the bivariate estimates were done for MM2. The SSE parameters were quite a bit different from the MLE's. The parameters are:

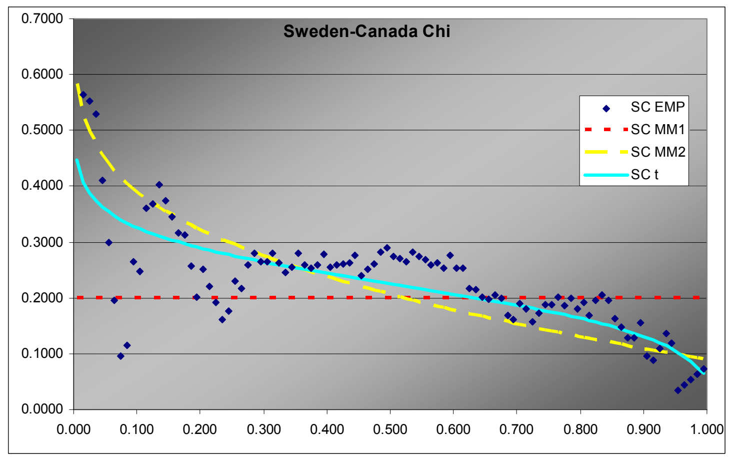
	MM2			**MM1**				**t**	
	SSE	**MLE bi**		**SSE**	**MLE bi**	**MLE tri**		**MLE bi**	**MLE tri**
δ_{12}	2.62588	1.50608		3.69513	2.14275	2.1094		0.491	0.490
δ_{13}	0.80055	0.43963		1.52005	1.19832	1.1152		0.262	0.266
δ_{23}	3.2E-07	0.0103		1	1	1		0.097	0.097
p_1	0.49881	0.37649		0.49963	0.40533	0.37175			
p_2	0.5	0.5		0.5	0.5	0.5			
p_3	0.26236	0.49625		0.29097	0.5	0.5			
θ	0.19599	0.2209		1.08308	1.0939	1.1234		20.53	20.95

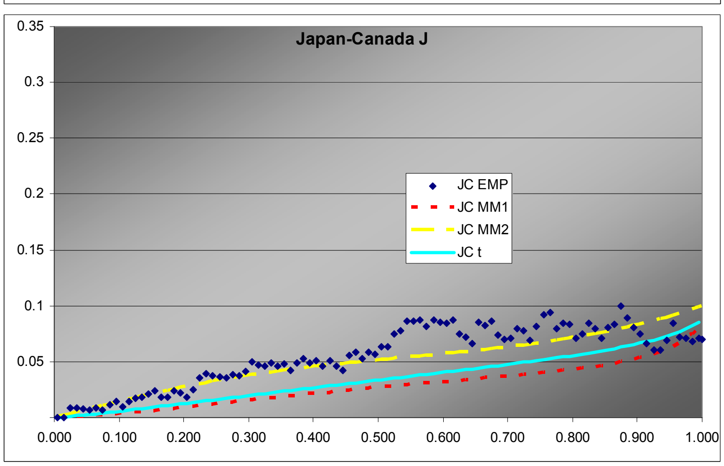
The t parameters were done trivariate and product of bivariates for comparison. We use a beta distribution version of the t that allows fractional degrees of freedom. Comparisons of fits were done using the J and χ functions. These are defined as $J(z) = -z^2 + 4\int_0^z\int_0^z C(u,v)c(u,v)dvdu/C(z,z)$ and $\chi(z) = 2 - \ln[C(z,z)]/\ln z$. As $z \to 1$ these approach Kendall's τ and the upper tail dependence, respectively.

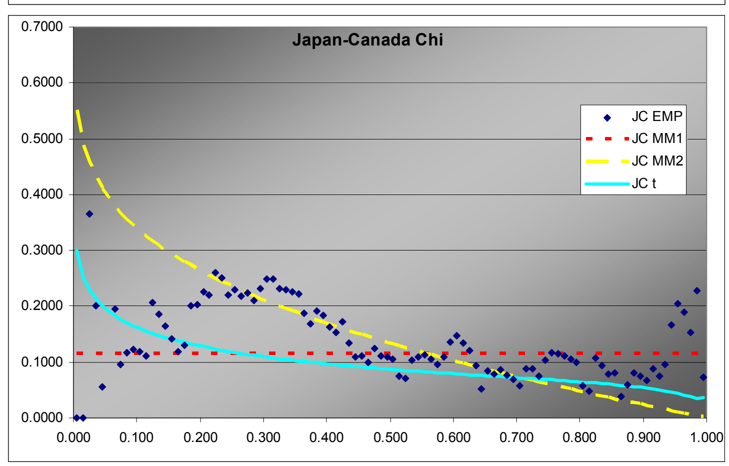












In two of the three J graphs, the t-copula is clearly the best fit, but MM2 is close for Sweden-Canada. In the third, MM2 is the best. MM1 is always worse for this data.

The t is also best in two of the three χ graphs, but not much better than MM2 for Sweden-Canada. In the third graph, each of the three is best in some range but MM1 is probably best overall.

SUMMARY
The MMj copulas do not give a lot of flexibility for a range of bivariate correlations. They would be most appropriate when the empirical correlations are all fairly close to each other. This gets more so as the number of dimensions increases. They all have somewhat different shapes, and differ from the t-copula as well, so could be useful with the right data. The MM3 is interesting in that the tail dependence increases as the correlation decreases.

Reference

1. Joe, H., *Multivariate Models and Dependence Concepts,* Chapman and Hall, 1997

Appendix – Numerical Density

0. Background & Notation
The discussion is oriented to copulas, i.e., smooth, parametric, cumulative distribution functions on the unit hypercube.
Let X ε Ω = $[0,1]^d$ with cdf F(X | θ). Sample points are denoted x alone or $x^{(i)}$ if a sequence needs to be indexed. Components of x are x_j.

1. From CDF to Measure
The hypercube H(x,δ) is defined as
$([x_1-\delta/2, x_1+\delta/2]\times[x_2-\delta/2, x_2+\delta/2]\times\ldots\times[x_d-\delta/2, x_d+\delta/2])\cap\Omega$.
Without loss of generality, we will assume a generic H is wholly contained in Ω.
We can identify the set of corners of H(x,δ) with the set ***S*** = $\{0,1\}^d$ by the mapping
$s=\langle s_1, s_2,\ldots,s_d\rangle \mapsto \langle x_1+(-1)^{s_1}\cdot\delta/2, x_2+(-1)^{s_2}\cdot\delta/2, \ldots, x_d+(-1)^{s_d}\cdot\delta/2\rangle = C(x,\delta,s)$.
Let $g(s)=(-1)^{\sum_j s_j}$ be the sign of *s*.
Define the probability measure μ(H | θ) as Pr{XεH | θ}.
Theorem: $\mu(H(x,\delta)|\theta)=\sum_{s\in\boldsymbol{S}} g(s)\cdot F(C(x,\delta,s)|\theta)$.

2. Approximating MLE
Since F is smooth, it has a probability density function defined by
$f(x|\theta)\overset{\Delta}{=}\lim_{\delta\to 0}\frac{\mu(H(x,\delta)|\theta)}{\delta^d}$. In general, this is intractable, i.e., a closed-form expression is not convenient. However, since μ can be evaluated, a numerical approximation

$f(x \mid \theta,\delta)$ for small δ is readily available by applying the theorem. Therefore, the log-likelihood of a set of data $\{x^{(1)},x^{(2)},\ldots,x^{(n)}\}$ can be approximated by

$$\mathbf{L}(\theta) \approx \sum_i \ln\left(f\left(x^{(i)} \mid \theta,\delta\right)\right)$$ for suitably chosen δ.

This allows the application of standard Maximum Likelihood Estimation techniques.

3. Sampling

While ML estimation carries with it its own asymptotic results for assessing standard errors, it might be useful to conduct simulation experiments to bolster these results. A form of rejection sampling is outlined here. Say the goal is to sample n points from the distribution defined by $F(\cdot \mid \theta)$.

Step 1: Choose $N >> n$ points $x^{(i)}$ uniformly from Ω.

Step 2: For each i, calculate $\phi_i = f(x^{(i)} \mid \theta,\delta)$. Let $m = \max\{\phi_i\}$ and $M = (\Sigma\phi_i)/m$. If $M<n$ then go back to step 1 and choose a larger N.

Step 3: Sample n points without replacement from the discrete distribution $\{< x^{(i)},p_i >\}$ where $p_i = n^*\phi_i/(\Sigma\phi_i)$. If you make sure the $x^{(i)}$ are in random order, e.g., the order in which they were first sampled, you can use the following systematic sampling procedure:

```
Sample := {}.  Sp := 0.  i := 0.
Repeat
   i:=i+1.
   If the interval (Sp, Sp+ p_i] contains an integer, then Sample := Sample ∪ {x^(i)}.
   Sp := Sp+ p_i.
Until Sp = n.
```

Bibliography

Tillé, Yves (1996) "Some Remarks on Unequal Probability Sampling Designs Without Replacement," *Annales D'Économie et de Statistique,* No. 44.
http://www.adres.polytechnique.fr/ANCIENS/n44/vol44-08.pdf